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André Deprit

J. F. Price

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André Deprit

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SUMMARY

The canonical equations of motion in barycentric synodical Cartesian coordinates and momenta are integrable by means of recurrent power series; these series are proved to be convergent for initial conditions anywhere in the phase space except in the two phase planes of binary collisions.

The integration by recurrent power series is extended to the variation equations. It is used to compute the monodromy matrix associated to the fundamental period of a periodic orbit. A simple formula is derived, which relates the trace of the monodromy matrix and the characteristic exponents.

These numerical methods are applied to evaluate the characteristic exponents of Rabe's Trojan Orbits; they are found to be of the stable type for the ovals, and of the unstable type for the horse-shoe shaped orbit.

When the periodic orbit is symmetric with respect to the axis of syzygies, four independent variational solutions computed only over half the period are shown to be sufficient for evaluating the characteristic exponents.

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1. INTRODUCTION

Steffensen (1956) has shown how the equations of motion for the planar Restricted Problem of Three Bodies lend themselves easily to the integration by recurrent power series in the time. He has set up the algorithm for the Lagrangian equations in the jovicentric synodical coordinate system, and he proved that, with the exception of initial conditions right at a binary collision, the power series are convergent.

These formulae, slightly modified on a minor point, were used extensively for the first time by Rabe (1961, 1962a, 1962b) in computing periodic Trojan orbits and long period ovals at L_4 in the Earth-Moon system. A similar algorithm for the Lagrangian equations in the barycentric synodical coordinate system has been proposed by Fehlberg (1964); he compared it with a Runge-Kutta-Nyström procedure. Even for an orbit of close approach to one of the primaries, Steffensen's method proved itself more accurate and time saving.

We propose here to adapt Steffensen's ideas to the canonical equations of motion in the barycentric synodical coordinate system. This implies that we replace this fourth order system by a system of eight differential equations, all of the first order. In the same manner as Steffensen did for the Lagrangian equations, we prove that the power series computed by recurrence are convergent for any set of

initial conditions, provided it does not belong to the phase planes $(x = -\mu,\ y = 0)$ or $(x = 1 - \mu,\ y = 0)$ of binary collisions.

Then we extend the method to the variational equations. These are linear equations whose coefficients are functions only of the coordinates along the reference orbit. For simplicity of presentation, we think of computing first these coefficients in power series by recurrence, using the power series representing the coordinates along the orbit; then the variational equations are integrated in turn by the recurrent power series method. We show how the computation can be controlled by the Jacobi integral to be verified by each variational solution; in the case when four independent variations are computed at the same time, a drastic check is provided by verifying at each step how the matrix of this fundamental system remains close to a completely canonical matrix.

Now that we are able to compute accurately and efficiently the matrizant (Danby 1964) along any orbit which is not on a collision course in the planar Restricted Problem of Three Bodies, we do not need to go through an approximate resolution of a second order differential equation of the Hill type (Darwin 1911, Message 1959, Rabe 1961) when it comes to computing the characteristic exponents of a periodic orbit. Indeed the characteristic roots are the eigenvalues of the matrizant at the end of the fundamental period, a matrix which is called by Wintner (1946) the *monodromy matrix*. Several elementary properties of

the matrix lead to a simple relation between its trace and the two non-trivial characteristic roots of the periodic orbit. The stability of a periodic orbit is characterized quite simply by this trace Tr(T): if $0 < Tr(T) < 4$, the characteristic exponents are of the stabie type; $Tr(T) = 0$ or $Tr(T) = 4$ give the two indifferent cases; in all other circumstances, the characteristic exponents are of the unstable type.

For the sake of completeness, we show how this method of computing the characteristic roots is simplified in the case of a periodic orbit which is symmetric with respect to the axis of syzygies. There we need to compute the matrizant over only half the period, and the characteristic roots are derived from the homomorphism axiom satisfied by the one-parameter group which the matrizant generates.

These numerical methods are tested on Rabe's Trojan Orbits. The initial conditions recorded by Rabe, the Jacobi constants and the periods have been converted from the jovicentric coordinate system and the units chosen by Darwin into the barycentric coordinate systems and the canonical units defined by Wintner. Then for those orbits which need it, especially the horse-shoe shaped orbit, the initial conditions have been improved. The characteristic roots are computed. For the oval-shaped orbits, our results confirm the indication which Rabe drew from a coarsely approximate solution of the Hill's equation in the variation normal to the orbit. However, for the horse-shoe shaped orbit,

whereas Rabe tentatively suggested a variational stability, we find variational instability. The disagreement has been confirmed in two ways: on our side, we recomputed Rabe's horse-shoe orbit by starting at a different point on the orbit, thus obtaining another matrizant, and we found this matrix to be equivalent, *modulo* a similitude, to the previously obtained matrizant. On his side, Schanzle (1965) recomputed the Fourier series expansion of the orbit and its associated Hill's equation; he found in its normal displacements far more oscillations than reported by Rabe. In fact, his analysis shows clearly that in this case, conclusions drawn from second order approximate solution of Hill's equation are just meaningless. Also, Schanzle applied the method devised by de Vogelaere (1950) and Brillouin (1948) for solving numerically Hill's equation, and he obtained thereby characteristic exponents of the unstable type.

2. EQUATIONS OF MOTION

The canonical units for mass, length and time are adopted as they are defined by A. Wintner (1946); the motion of the planetoid is referred to the barycentric synodical coordinate system. Thus the planar Restricted Problem of Three Bodies is described by the Hamiltonian function

$$\mathcal{H} = \tfrac{1}{2}(p_x^2 + p_y^2) - (xp_y - yp_x) - \frac{1-\mu}{\rho_1} - \frac{\mu}{\rho_2} \tag{1}$$

where

$$(2a)\qquad \rho_1 = |(x+\mu)^2 + y^2|^{\frac{1}{2}},$$

$$(2b)\qquad \rho_2 = |(x+\mu-1)^2 + y^2|^{\frac{1}{2}}.$$

The canonical equations of motion

$$(3)\qquad \begin{cases} \dot{x} = p_x + y, \\ \dot{y} = p_y - x, \\ \dot{p}_x = -\left(\dfrac{1-\mu}{\rho_1^3} + \dfrac{\mu}{\rho_2^3}\right)x + p_y - \mu\dfrac{1-\mu}{\rho_1^3} + (1-\mu)\dfrac{\mu}{\rho_2^3}, \\ \dot{p}_y = -\left(\dfrac{1-\mu}{\rho_1^3} + \dfrac{\mu}{\rho_2^3}\right)y - p_x \end{cases}$$

admit the first integral

$$(4)\qquad C = -\mu(\mu - 1) - 2\mathcal{H};$$

this Jacobi constant C is so chosen that $C = 3$ at the equilateral equilibrium configuration whatever the mass ratio μ may be.

With the introduction of the quantities

$$(5)\qquad R = (1-\mu)\rho_1^{-3}, \qquad S = \mu\rho_2^{-3},$$

the canonical equations (3) may be replaced by a system of eight differential equations

$$\dot{x} = p_x + y,$$

$$\dot{y} = p_y - x,$$

$$\rho_1\dot{\rho}_1 = x\dot{x} + y\dot{y} + \mu\dot{x},$$

$$\rho_2\dot{\rho}_2 = x\dot{x} + y\dot{y} - (1 - \mu)\dot{x},$$

$$\rho_1\dot{R} = -3R\dot{\rho}_1,$$

$$\rho_2\dot{S} = -3S\dot{\rho}_2,$$

$$\dot{p}_x = -(R + S)x + p_y - \mu R + (1 - \mu)S,$$

$$\dot{p}_y = -(R + S)y - p_x$$

which lend themselves in an obvious way to an integration by recurrent power series.

The formal power series

$$x = \sum_{n \geq 0} x_n(\Delta t)^n, \qquad \rho_1 = \sum_{n \geq 0} r_n(\Delta t)^n,$$

$$y = \sum_{n \geq 0} y_n(\Delta t)^n, \qquad \rho_2 = \sum_{n \geq 0} s_n(\Delta t)^n,$$

$$p_x = \sum_{n \geq 0} p_n(\Delta t)^n, \qquad R = \sum_{n \geq 0} R_n(\Delta t)^n,$$

$$p_y = \sum_{n \geq 0} q_n(\Delta t)^n, \qquad S = \sum_{n \geq 0} S_n(\Delta t)^n$$

are introduced into the differential equations and coefficients of $(\Delta t)^n$ are collected together for each $n=0,1,2,\ldots$. In this manner, for each n

one obtains the eight relations

$$(n+1)x_{n+1} = p_n + y_n,$$

$$(n+1)y_{n+1} = q_n - x_n,$$

$$\sum_{0 \le p \le n} (p+1)r_{p+1}r_{n-p} = \sum_{0 \le p \le n} (p+1)(x_{p+1}x_{n-p} + y_{p+1}y_{n-p}) + (n+1)\mu x_{n+1},$$

$$\sum_{0 \le p \le n} (p+1)s_{p+1}s_{n-p} = \sum_{0 \le p \le n} (p+1)(x_{p+1}x_{n-p} + y_{p+1}y_{n-p}) - (n+1)(1-\mu)x_{n+1},$$

$$\sum_{0 \le p \le n} (p+1)(3r_{p+1}R_{n-p} + r_{n-p}R_{p+1}) = 0$$

$$\sum_{0 \le p \le n} (p+1)(3s_{p+1}S_{n-p} + s_{n-p}S_{p+1}) = 0,$$

$$(n+1)p_{n+1} = - \sum_{0 \le p \le n} x_p(R_{n-p} + S_{n-p}) + q_n - \mu R_n + (1-\mu)S_n,$$

$$(n+1)q_{n+1} = - \sum_{0 \le p \le n} y_p(R_{n-p} + S_{n-p}) - p_n.$$

Initial conditions evidently give

$$x_0 = x(0), \qquad y_0 = y(0), \qquad p_0 = p_x(0), \qquad q_0 = p_y(0),$$

the definition of the additional unknowns require that

$$r_0 = |(x_0 + \mu)^2 + y_0^2|^{\frac{1}{2}}, \qquad R_0 = (1 - \mu)r_0^{-3},$$

$$s_0 = |(x_0 + \mu - 1)^2 + y_0^2|^{\frac{1}{2}}, \qquad S_0 = \mu s_0^{-3}.$$

Once the coefficients of degree 0 in t are determined, those of first degree are computed from the formulas:

$$x_1 = p_0 + y_0,$$

$$y_1 = q_0 - x_0,$$

$$r_0 r_1 = (x_0 x_1 + y_0 y_1) + \mu x_1,$$

$$s_0 s_1 = (x_0 x_1 + y_0 y_1) - (1 - \mu) x_1,$$

$$r_0 R_1 = -3 r_1 R_0,$$

$$s_0 S_1 = -3 s_1 S_0,$$

$$p_1 = -x_0 (R_0 + S_0) + q_0 - \mu R_0 - (1 - \mu) S_0,$$

$$q_1 = -y_0 (R_0 + S_0) - p_0.$$

The recurrence steps from degree n (≥ 1) to degree $n + 1$ are given explicitly by means of the formulas

(6a) $(n+1) x_{n+1} = p_n + y_n,$

(6b) $(n+1) y_{n+1} = q_n - x_n,$

(6c) $$(n+1) r_0 r_{n+1} = \sum_{0 \leq p \leq n-1} (p+1)(x_{p+1} x_{n-p} + y_{p+1} y_{n-p} - r_{p+1} r_{n-p}) + (n+1)(\mu x_{n+1} + x_0 x_{n+1} + y_0 y_{n+1}),$$

(6d) $$(n+1) s_0 s_{n+1} = \sum_{0 \leq p \leq n-1} (p+1)(x_{p+1} x_{n-p} + y_{p+1} y_{n-p} - s_{p+1} s_{n-p}) + (n+1)(x_0 x_{n+1} + y_0 y_{n+1} - (1-\mu) x_{n+1}),$$

$$(6e)\quad (n+1)r_0R_{n+1} = -\sum_{0\le p\le n-1}(p+1)(3r_{p+1}R_{n-p}+r_{n-p}R_{p+1})-3(n+1)r_{n+1}R_0,$$

$$(6f)\quad (n+1)s_0S_{n+1} = -\sum_{0\le p\le n-1}(p+1)(3s_{p+1}S_{n-p}+s_{n-p}S_{p+1})-3(n+1)s_{n+1}S_0,$$

$$(6g)\quad (n+1)p_{n+1} = -\sum_{0\le p\le n}x_p(R_{n-p}+S_{n-p})+q_n-\mu R_n+(1-\mu)S_n,$$

$$(6h)\quad (n+1)q_{n+1} = -\sum_{0\le p\le n}y_p(R_{n-p}+S_{n-p})-p_n.$$

If the coefficients at each step are computed in the order in which the above formulas have been written down, only known quantities will occur on the right hand side of the equations.

In order to prove that, when $r_0s_0 \neq 0$, the power series are convergent, we introduce, for every $n \geq 1$, the notation

$$k_n = \frac{1}{n(n+1)}$$

and we show that, for every $n \geq 2$, the inequalities

$$|x_n| \leq \bar{x}k_n\epsilon^n, \qquad |r_n| \leq \bar{r}k_n\epsilon^n,$$

$$|y_n| \leq \bar{y}k_n\epsilon^n, \qquad |s_n| \leq \bar{s}k_n\epsilon^n,$$

$$|p_n| \leq \bar{p}k_n\epsilon^n, \qquad |R_n| \leq \bar{R}k_n\epsilon^n,$$

$$|q_n| \leq \bar{q}k_n\epsilon^n, \qquad |S_n| \leq \bar{S}k_n\epsilon^n,$$

imply the inequalities

(7a) $|x_{n+1}| \leq \bar{x}k_{n+1}\varepsilon^{n+1}$, (7c) $|r_{n+1}| \leq \bar{r}k_{n+1}\varepsilon^{n+1}$,

(7b) $|y_{n+1}| \leq \bar{y}k_{n+1}\varepsilon^{n+1}$, (7d) $|s_{n+1}| \leq \bar{s}k_{n+1}\varepsilon^{n+1}$,

(7g) $|p_{n+1}| \leq \bar{p}k_{n+1}\varepsilon^{n+1}$, (7e) $|R_{n+1}| \leq \bar{R}k_{n+1}\varepsilon^{n+1}$,

(7h) $|q_{n+1}| \leq \bar{q}k_{n+1}\varepsilon^{n+1}$, (7f) $|S_{n+1}| \leq \bar{S}k_{n+1}\varepsilon^{n+1}$.

Dealing first with (6a), we obtain

$$(n + 1)|x_{n+1}| \leq (\bar{p} + \bar{y})k_n\varepsilon^n,$$

hence a sufficient condition for the validity of (7a) is that

$$(\bar{p} + \bar{y})k_n\varepsilon^n \leq (n + 1)\bar{x}k_{n+1}\varepsilon^{n+1}.$$

Since

$$\frac{k_n}{(n+1)k_{n+1}} = \frac{n+2}{n(n+1)} = \frac{1}{n+1}\left(1 + \frac{2}{n}\right) \leq \frac{2}{3} \qquad \text{for} \quad n \geq 2,$$

the more rigid inequality

(8a) $$\frac{2}{3}(\bar{p} + \bar{y}) \leq \varepsilon\bar{x}$$

is also a sufficient condition for the validity of (7a).

Treating (6b) in the same way, we obtain as a sufficient condition for (7b) that

$$\text{(8b)} \qquad \frac{2}{3}(\bar{q} + \bar{x}) \leq \varepsilon\bar{y}.$$

From (6c), we obtain as a sufficient condition for (7c) that

$$(n+2)(\bar{x}^2+\bar{y}^2+\bar{r}^2) \sum_{0 \leq p \leq n-1} (p+1)k_{p+1}k_{n-p} + (\mu+|x_0|)\bar{x} + |y_0|\bar{y} \leq r_0\bar{r}.$$

But from the relation

$$n + 2 = (p + 2)(n - p + 1) - (p + 1)(n - p),$$

we deduce that

$$(n+2)k_{p+1}k_{n-p} = \frac{1}{n+1}\left(\frac{1}{p+1} + \frac{1}{n-p}\right) - \frac{1}{n+3}\left(\frac{1}{p+2} + \frac{1}{n-p+1}\right),$$

hence that

$$(n+2) \sum_{0 \leq p \leq n-1} (p+1)k_{p+1}k_{n-p} = \frac{n+2\sigma_n}{n+3}$$

where we have defined, for any $n \geq 1$,

$$\sigma_n = \sum_{1 \leq p \leq n} \frac{1}{p} .$$

However, for any $n \geq 2$,

$$\sigma_n \leq 1 + \tfrac{1}{2} + \frac{n-2}{3} = \frac{3}{2} + \frac{n-2}{3}$$

so that

$$n + 2\sigma_n \leq \frac{5}{3}(n+1)$$

and

$$\frac{n + 2\sigma_n}{n + 3} \leq \frac{5}{3}(1 - \frac{2}{n + 3}) \leq 5/3.$$

Consequently, (7c) is verified if

$$\text{(8c)} \qquad \frac{5}{3}(\bar{x}^2 + \bar{y}^2) + (\mu + |x_0|)\bar{x} + |y_0|\bar{y} \leq (r_0 - \frac{5}{3}\bar{r})\bar{r}.$$

By symmetry, from (6d) we obtain as a sufficient condition for (7d) that

$$\text{(8d)} \qquad \frac{5}{3}(\bar{x}^2 + \bar{y}^2) + ((1 - \mu) + |x_0|)\bar{x} + |y_0|\bar{y} \leq (s_0 - \frac{5}{3}\bar{s})\bar{s}.$$

We now address ourselves to (6e). Using the same relations and the same estimates as in the preceding case, we find that

$$\text{(8e)} \qquad 3R_0\bar{r} \leq (r_0 - \frac{20}{3}\bar{r})\bar{R}$$

is a sufficient condition for (7e), and by symmetry that

$$\text{(8f)} \qquad 3S_0\bar{s} \leq (s_0 - \frac{20}{3}\bar{s})\bar{S}$$

is a sufficient condition for (7f).

At last we examine (6g). We give it the form

$$(n+1)p_{n+1} = -x_0(R_n+S_n) - x_n(R_0+S_0) + q_n - \mu R_n + (1-\mu)S_n - \sum_{1 \leq p \leq n-1} x_p(R_{n-p}+S_{n-p})$$

so that we come to the inequality

$$(n+1)|p_{n+1}| \leq k_n \varepsilon^n[(\bar{R} + \bar{S})|x_0| + (R_0 + S_0)\bar{x} + \bar{q} + \mu\bar{R} + (1 - \mu)\bar{S}]$$

$$+ \bar{x}(\bar{R}+\bar{S})\varepsilon^n \sum_{1 \leq p \leq n-1} k_p k_{n-p}.$$

Therefrom we deduce that a sufficient condition for (7g) is the inequality

$$k_n[(R+S)\ x_0 + (R_0+S_0)\bar{x} + \bar{q} + \mu\bar{R} + (1-\mu)\bar{S}] + \bar{x}(\bar{R}+\bar{S}) \sum_{1 \leq p \leq n-1} k_p k_{n-p} \leq (n+1)k_{n+1}\bar{p}\varepsilon .$$

But the identity

$$n + 1 = (p+1)(n-p+1) - p(n-p)$$

implies that

$$(n+1)k_p k_{n-p} = \frac{1}{n}(\frac{1}{p} + \frac{1}{n-p}) - \frac{1}{n+2}(\frac{1}{p+1} + \frac{1}{n-p+1})$$

and hence that

$$(n+1) \sum_{1 \leq p \leq n-1} k_p k_{n-p} = \frac{2(n-1+2\sigma_{n-1})}{n(n+2)} .$$

However,

$$\sigma_{n-1} \leq n - 1,$$

which implies that

$$\frac{n-1+2\sigma_{n-1}}{n} \leq 3(1 - \frac{1}{n}) < 3.$$

Therefore, a sufficient condition for (7g) is that

$$\text{(8g)} \qquad \frac{2}{3}[(\bar{R} + \bar{S})|x_0| + (R_0 + S_0)\bar{x} + \bar{q} + \mu\bar{R} + (1 - \mu)\bar{S}] + 3\bar{x}(\bar{R}+\bar{S}) \leq \bar{p}\varepsilon$$

By symmetry from (6h), a sufficient condition for (7h) is that

$$\text{(8h)} \qquad \frac{2}{3}[(\bar{R} + \bar{S})|y_0| + (R_0 + S_0)\bar{y}] + 3\bar{x}(\bar{R}+\bar{S}) \leq \bar{q}\varepsilon.$$

Now that we found sufficient conditions to be fulfilled in order that the inequalities should be fulfilled recurrently, we have to check that they are compatible.

To begin with, ε can always be chosen so large that (8a), (8b), (8g) and (8h) are satisfied no matter what values the constants possess. Also, it follows from (8e) and (8f) that we must choose

$$\bar{r} < 3r_0/20, \qquad \bar{s} < 3s_0/20,$$

after which (8e) and (8f) are satisfied provided that we choose $\bar{R}$ and $\bar{S}$ sufficiently large. After this, (8c) and (8d) will be satisfied, if we choose $\bar{x}$ and $\bar{y}$ sufficiently small in comparison with $\bar{r}$ and $\bar{s}$. In thus choosing small values for $\bar{x}$ and $\bar{y}$, we do not run into difficulties, because the inequalities (7) show that small values of these constants can

be compensated by choosing ε sufficiently large.

The inequalities (7) being satisfied for any $n \geq 2$, the recurrent series (6) are dominated each by series of the form

$$A + Bt + C \sum_{n \geq 2} k_n \varepsilon^n t^n ,$$

which is convergent in the disk $|t| < 1/\varepsilon$. Thus the series (6) are convergent in this disk.

3. VARIATION EQUATIONS

We denote by the vector δ the displacements $(\delta x, \delta y, \delta p_x, \delta p_y)$ of a solution $t \to (x(t), y(t), p_x(t), p_y(t))$ of the canonical equations (3). This vector is determined by the vector variation equation

$$\dot{\delta} = V(t)\delta. \tag{9}$$

$V(t)$ is a 4×4 matrix function of the form

$$V(t) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ \alpha(t) & \beta(t) & 0 & 1 \\ \beta(t) & \gamma(t) & -1 & 0 \end{pmatrix} \tag{10}$$

wherein

$$\alpha(t) = -\frac{1-\mu}{\rho_1^3(t)} \left[1 - 3\,\frac{(x(t)+\mu)^2}{\rho_1^2(t)} \right] - \frac{\mu}{\rho_2^3(t)} \left[1 - 3\,\frac{(x(t)+\mu-1)^2}{\rho_2^2(t)} \right], \tag{11a}$$

(11b) $$\beta(t) = 3\,\frac{1-\mu}{\rho_1^3(t)}\,\frac{(x(t)+\mu)y(t)}{\rho_1^2(t)} + 3\,\frac{\mu}{\rho_2^3(t)}\,\frac{(x(t)+\mu-1)y(t)}{\rho_2^2(t)},$$

(11c) $$\gamma(t) = -\frac{1-\mu}{\rho_1^3(t)}\left[1 - 3\,\frac{y^2(t)}{\rho_1^2(t)}\right] - \frac{\mu}{\rho_2^3(t)}\left[1 - 3\,\frac{y^2(t)}{\rho_2^2(t)}\right].$$

The variational equations (9) are the canonical system derived from the Hamiltonian function

(12) $$\mathcal{V} = \tfrac{1}{2}(\delta p_x^2 + \delta p_y^2) - (\delta x\delta p_y - \delta y\delta p_x) - \tfrac{1}{2}[\alpha(t)\delta x^2 + 2\beta(t)\delta x\delta y + \gamma(t)\delta y^2].$$

Because the original Hamiltonian function (1) is conservative, the equations (9) verify the Jacobi variational integral

(13) $$\Gamma = \delta x\cdot\frac{\partial\mathcal{H}}{\partial x} + \delta y\cdot\frac{\partial\mathcal{H}}{\partial y} + \delta p_x\cdot\frac{\partial\mathcal{H}}{\partial p_x} + \delta p_y\cdot\frac{\partial\mathcal{H}}{\partial p_y}$$

where the coordinates and momenta in the partial derivatives should be given their values at each time along the orbit.

It is our purpose to show that the variational equations can be integrated by recurrent power series together with the equations of motion.

For simplicity we think of our task as two-fold. At each step of the recurrence, we first compute the coefficients in the power series representing α, β and γ. Then by means of the variational equations, we compute the

corresponding coefficients in the power series representing the displacements.

The recurrent power series expansion of α, β, γ requires several auxiliary variables. The selection may vary widely according to one's prejudices and predilections in programming; it might also be influenced by the kind of mathematical information one would like to draw on the side from the integration. We present here a list yielding fairly elegant recurrence formulas. Our own computer program actually uses a list with four less auxiliary variables needed. Here we introduce in a first block

$$A = \frac{x + \mu}{\rho_1}, \qquad B = \frac{x + \mu - 1}{\rho_2}, \qquad C = \frac{y}{\rho_1}, \qquad D = \frac{y}{\rho_2},$$

and in a second block

$$E = 1 - 3\,A^2, \qquad G = AC, \qquad J = 1 - 3\,C^2,$$

$$F = 1 - 3\,B^2, \qquad H = BD, \qquad K = 1 - 3\,D^2,$$

so that the time dependent coefficients in the matrix V take the simple form

$$\alpha = -(RE + SF),$$

$$\beta = 3(RG + SH),$$

$$\gamma = -(RJ + SK).$$

We denote the coefficients in their power series in the natural way:

$$A = \sum_{n \geq 0} A_n (\Delta t)^n, \quad \text{and so on.}$$

As for the variations themselves, we have put

$$\delta x = \sum_{n > 0} \xi_n (\Delta t)^n, \qquad \delta p_x = \sum_{n \geq 0} \phi_n (\Delta t)^n,$$

$$\delta v = \sum_{n \geq 0} \eta_n (\Delta t)^n, \qquad \delta p_y = \sum_{n \geq 0} \psi_n (\Delta t)^n.$$

For $n = 0$, the coefficients in the auxiliary series A to K and in the functions, α, β, γ are computed from the initial conditions on the orbit, while they are determined in the variations from the chosen initial values for the displacement. Once the coefficients of degree n have been computed for the coordinates and momenta along the orbit, and the displacements to the orbit, the coefficients of degree $n + 1$ are computed by means of the formulae (6) to be followed by four new sets of formulae. The set that ought to be handled first is

(14a) $$r_0 A_n = x_n - \sum_{0 \leq p \leq n-1} r_{n-p} A_p,$$

(14b) $$s_0 B_n = x_n - \sum_{0 \leq p \leq n-1} s_{n-p} B_p,$$

(14c) $$r_0 C_n = y_n - \sum_{0 \leq p \leq n-1} r_{n-p} C_p,$$

(14d) $$s_0 D_n = y_n - \sum_{0 \leq p \leq n-1} s_{n-p} D_p;$$

then, the recurrence should go through the formulae

(15a) $E_n = -3 \sum_{0 \le p \le n} A_p A_{n-p}$, (15b) $F_n = -3 \sum_{0 \le p \le n} B_p B_{n-p}$,

(15c) $G_n = \sum_{0 \le p \le n} A_p C_{n-p}$, (15d) $H_n = \sum_{0 \le p \le n} B_p D_{n-p}$,

(15c) $J_n = -3 \sum_{0 \le p \le n} C_p C_{n-p}$, (15f) $K_n = -3 \sum_{0 \le p \le n} D_p D_{n-p}$,

before the coefficients of the time functions in the matrix V could be computed by

(16a) $\alpha_n = - \sum_{0 \le p \le n} (R_p E_{n-p} + S_p F_{n-p})$,

(16b) $\beta_n = 3 \sum_{0 \le p \le n} (R_p G_{n-p} + S_p H_{n-p})$,

(16c) $\gamma_n = - \sum_{0 \le p \le n} (R_p J_{n-p} + S_p K_{n-p})$.

Finally the coefficients of degree n in the variational solutions are given by

(17a) $(n + 1)\xi_{n+1} = \eta_n + \phi_n$,

(17b) $(n + 1)\eta_{n+1} = -\xi_n +$ [illegible]$_n$,

$$(17c)\quad (n+1)\varphi_{n+1} = \psi_n + \sum_{0 \le p \le n} (\alpha_p \xi_{n-p} + \beta_p \eta_{n-p}),$$

$$(17d)\quad (n+1)\psi_{n+1} = -\varphi_n + \sum_{0 \le p \le n} (\beta_p \xi_{n-p} + \gamma_p \eta_{n-p}).$$

A proof of the convergence for the series in the variations follows the same lines as the proof we gave in the first section of this paper.

4. CHARACTERISTIC EXPONENTS

We consider the four solutions $\delta^{I}, \delta^{II}, \delta^{III}, \delta^{IV}$ of the variational equations which are determined respectively by the initial conditions

$$\begin{array}{llll}
\delta x^{I}(0) = 1, & \delta y^{I}(0) = 0, & \delta p_x^{I}(0) = 0, & \delta p_y^{I}(0) = 0, \\
\delta x^{II}(0) = 0, & \delta y^{II}(0) = 1, & \delta p_x^{II}(0) = 0, & \delta p_y^{II}(0) = 0, \\
\delta x^{III}(0) = 0, & \delta y^{III}(0) = 0, & \delta p_x^{III}(0) = 1, & \delta p_y^{III}(0) = 0, \\
\delta x^{IV}(0) = 0, & \delta y^{IV}(0) = 0, & \delta p_x^{IV}(0) = 0, & \delta p_y^{IV}(0) = 1.
\end{array}$$

We call $R(t;0)$ the matrix whose columns are made of these four solutions; this is nothing else than the matrizant of the variational equations such that

$$R(0;0) = I_4$$

(I_4 denotes the $4 \cdot 4$ unit matrix.)

The fact that, for any t, $R(t)$ is a symplectic matrix, provides another check on the accuracy of the numerical integration which produced the four fundamental solutions. Indeed, the matrix identity

$$R(t;0)J(R(t;0))^T = J,$$

where

$$J = \begin{pmatrix} 0_2 & I_2 \\ -I_2 & 0_2 \end{pmatrix},$$

is equivalent to the 6 independent bilinear identities

(19a) $\delta x^I(t)\delta y^{III}(t) - \delta x^{III}(t)\delta y^I(t) + \delta x^{II}(t)\delta y^{IV}(t) - \delta x^{IV}(t)\delta y^{II}(t) = 0,$

(19b) $\delta x^I(t)\delta p_x^{III}(t) - \delta x^{III}(t)\delta p_x^I(t) + \delta x^{II}(t)\delta p_x^{IV}(t) - \delta x^{IV}(t)\delta p_x^{II}(t) = 1,$

(19c) $\delta x^I(t)\delta p_y^{III}(t) - \delta x^{III}(t)\delta p_y^I(t) + \delta x^{II}(t)\delta p_y^{IV}(t) - \delta x^{IV}(t)\delta p_y^{II}(t) = 0,$

(19d) $\delta y^I(t)\delta p_x^{III}(t) - \delta y^{III}(t)\delta p_x^I(t) + \delta y^{II}(t)\delta p_x^{IV}(t) - \delta y^{IV}(t)\delta p_x^{II}(t) = 0,$

(19e) $\delta y^I(t)\delta p_y^{III}(t) - \delta y^{III}(t)\delta p_y^I(t) + \delta y^{II}(t)\delta p_y^{IV}(t) - \delta y^{IV}(t)\delta p_y^{II}(t) = 1,$

(19f) $\delta p_x^I(t)\delta p_y^{III}(t) - \delta p_x^{III}(t)\delta p_y^I(t) + \delta p_x^{II}(t)\delta p_y^{IV}(t) - \delta p_x^{IV}(t)\delta p_y^{II}(t) = 0.$

It should be emphasized that the check through these identities is more stringent than the one provided by the variational Jacobi integrals. Moreover, it dispenses with checking that $\det(R(t;0)) = 1$. For in a phase space whose dimension is even, the fact that the matrix $R(t;0)$ is completely canonical

implies that its determinant is equal to unity. Besides, as it has been shown by Bennett (1965), the matrix $R(t;0)$ is usually ill-conditioned so that the computation of its determinant may become meaningless, although it still makes sense to compute linear combinations of its minors of order 2.

The characteristic roots associated with the fundamental period T of a periodic orbit are the roots of the polynomial equation

$$\det\,(R(T;0) - sI_4) = 0. \tag{20}$$

It is known that it has at least two roots equal to $+1$, and that the other two roots s_1 and s_2 are such that

$$s_1 s_2 = +1.$$

We put

$$s_1 = je^{\Omega T}, \qquad s_2 = je^{-\Omega T}$$

with $j = \pm 1$; Ω is called the *characteristic exponent* associated to the period T.

But in the characteristic equation, the sum of the roots is equal to the trace, denoted here $Tr(T)$, of the matrix $R(T;0)$; thus

$$Tr(T) = \delta x^{I}(T) + \delta y^{II}(T) + \delta p_x^{III}(T) + \delta p_y^{IV}(T)$$

and

$$2 + 2j \cosh \Omega T = Tr(T). \tag{21}$$

This is a simple expression relating the characteristic exponent and the trace of the monodromy matrix.

Table I summarizes the discussion concerning the variational stability that ensues from the fundamental relation (21).

TABLE I. VARIATIONAL STABILITY OF A PERIODIC ORBIT AS DEFINED BY THE TRACE OF ITS MONODROMY MATRIX

	Ω	j	
$Tr(T) > 4$	real	+1	even instability
$Tr(T) = 4$	0	+1	indifferent case
$2 \leq Tr(T) < 4$	purely imaginary	+1	even stability
$0 < Tr(T) \leq 2$	purely imaginary	-1	odd stability
$Tr(T) = 0$	0	-1	indifferent case
$Tr(T) < 0$	real	-1	odd instability

5. CHARACTERISTIC EXPONENTS OF SYMMETRIC ORBITS

Along a periodic orbit, the matrix $V(t)$ in the right-hand member of the variation equations is a periodic function. Thus the identity

$$\dot{R}(t;0) = V(t)R(t;0)$$

implies the identity

$$\dot{R}(t + T;0) = V(t)R(t + T;0).$$

Accordingly, since $R(t;0)$ is the matrizant of the variation equations, we have the identity

$$R(t + T;0) = R(t;0)R(T;0);$$

in particular, at $t = -T/2$,

$$R(T/2;0) = R(-T/2;0)R(T;0).$$

Therefore

$$R(-T/2;0)[R(T;0) - sI_4] = R(T/2;0) - sR(-T/2;0),$$

and this proves that the characteristic equation (20) is equivalent to the equation

$$\det[R(T/2;0) - sR(-T/2;0)] = 0. \tag{22}$$

The equivalence just stated is valid for any periodic orbit, whether symmetric or asymmetric.

We shall now consider the particular case of a symmetric periodic orbit.

In order that the orbit $t \to (x(t),y(t),p_x(t),p_y(t))$ be symmetric with respect to the syzygy axis Ox, it is necessary and sufficient that, at any time,

$$x(-t) = +\, x(t), \qquad y(-t) = -y(t).$$

Consequently, along such an orbit,

$$\alpha(-t) = \alpha(t),$$

$$\beta(-t) = -\beta(t),$$

$$\gamma(-t) = \gamma(t).$$

Thus, if we put

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

the fact that the orbit is symmetric with respect to Ox implies in the variation equations that

$$V(-t) = -SV(t)S.$$

Accordingly, the substitution $t \to -t$ in the identity

$$\dot{R}(t;0) = V(t)R(t;0)$$

provides the identity

$$\dot{R}(-t;0) = SV(t)SR(-t;0)$$

or, since $S^2 = I_4$, the identity

$$\frac{d}{dt}[SR(-t;0)] = V(t)[SR(-t;0)].$$

Hence, using again the fact that $R(t;0)$ is the matrizant of the variation equations, we come at last to the identity

$$(23) \qquad SR(-t;0) = R(t;0)S.$$

In particular, at time $t = T/2$,

$$SR(-T/2;0) = R(T/2;0)S.$$

Therefore, the determinantal equation (20) takes the form

$$(24) \qquad \det[R(T/2;0) - sSR(T/2;0)S] = 0;$$

it proves that the computation of the characteristic exponents for a symmetric orbit requests the integration of the variational equations over only half a period.

This proposition has been stated first by Moulton (1914); but the proof he gives for it depends too closely on the particular problem he is dealing with, namely the orbital stability of Jupiter's satellite VIII, and it is incorrect on several points. de Vogelaere (1950) has shown how to use it for extracting numerically from Hill's equation the characteristic exponents of a symmetric orbit. We now propose to do the same for the solution of equation (24).

First we observe that the matrix identity (23) is equivalent to the scalar identities:

$$\begin{aligned}
\delta x^{I}(-t) &= \delta x^{I}(t), & \delta x^{II}(-t) &= -\delta x^{II}(t), \\
\delta y^{I}(-t) &= -\delta y^{I}(t), & \delta y^{II}(-t) &= \delta y^{II}(t), \\
\delta p_x^{I}(-t) &= -\delta p_x^{I}(t), & \delta p_x^{II}(-t) &= \delta p_x^{II}(t), \\
\delta p_y^{I}(-t) &= \delta p_y^{I}(t), & \delta p_y^{II}(-t) &= -\delta p_y^{II}(t),
\end{aligned}$$

$$\begin{aligned}
\delta x^{III}(-t) &= -\delta x^{III}(t), & \delta x^{IV}(-t) &= \delta x^{IV}(t), \\
\delta y^{III}(-t) &= \delta y^{III}(t), & \delta y^{IV}(-t) &= -\delta y^{IV}(t), \\
\delta p_x^{III}(-t) &= \delta p_x^{III}(t), & \delta p_x^{IV}(-t) &= -\delta p_x^{IV}(t), \\
\delta p_y^{III}(-t) &= -\delta p_y^{III}(t), & \delta p_y^{IV}(-t) &= \delta p_y^{IV}(t).
\end{aligned}$$

Therefore, the determinantal equation (24) can be written explicitly as

$$(25)\quad \begin{vmatrix}
(1-s)\delta x^{I}(T/2) & (1+s)\delta x^{II}(T/2) & (1+s)\delta x^{III}(T/2) & (1-s)\delta x^{IV}(T/2) \\
(1+s)\delta y^{I}(T/2) & (1-s)\delta y^{II}(T/2) & (1-s)\delta y^{III}(T/2) & (1+s)\delta y^{IV}(T/2) \\
(1+s)\delta p_x^{I}(T/2) & (1-s)\delta p_x^{II}(T/2) & (1-s)\delta p_x^{III}(T/2) & (1+s)\delta p_x^{IV}(T/2) \\
(1-s)\delta p_y^{I}(T/2) & (1+s)\delta p_y^{II}(T/2) & (1+s)\delta p_y^{III}(T/2) & (1-s)\delta p_y^{IV}(T/2)
\end{vmatrix} = 0.$$

We put

$$(1,2)(3,4)=[\delta x^{I}(T/2)\delta y^{IV}(T/2)-\delta x^{IV}(T/2)\delta y^{I}(T/2)]$$
$$[\delta p_x^{II}(T/2)\delta p_y^{III}(T/2)-\delta p_x^{III}(T/2)\delta p_y^{II}(T/2)],$$

$$(1,3)(2,4)=[\delta x^{I}(T/2)\delta p_x^{IV}(T/2)-\delta x^{IV}(T/2)\delta p_x^{I}(T/2)]$$

$$[\delta y^{II}(T/2)\delta p_y^{III}(T/2)-\delta y^{III}(T/2)\delta p_y^{II}(T/2)],$$

$$(1,4)(2,3)=[\delta x^{I}(T/2)\delta p_y^{IV}(T/2)-\delta x^{IV}(T/2)\delta p_y^{I}(T/2)]$$

$$[\delta y^{II}(T/2)\delta p_x^{III}(T/2)\delta y^{III}(T/2)\delta p_x^{II}(T/2)],$$

$$(2,3)(1,4)=[\delta y^{I}(T/2)\delta p_x^{IV}(T/2)-\delta y^{IV}(T/2)\delta p_x^{I}(T/2)]$$

$$[\delta x^{II}(T/2)\delta p_y^{III}(T/2)-\delta x^{III}(T/2)\delta p_y^{II}(T/2)],$$

$$(2,4)(1,3)=[\delta y^{I}(T/2)\delta p_y^{IV}(T/2)-\delta y^{IV}(T/2)\delta p_y^{I}(T/2)]$$

$$[\delta x^{II}(T/2)\delta p_x^{III}(T/2)-\delta x^{III}(T/2)\delta p_x^{II}(T/2)],$$

$$(3,4)(1,2)=[\delta p_x^{I}(T/2)\delta p_y^{IV}(T/2)-\delta p_x^{IV}(T/2)\delta p_y^{I}(T/2)]$$

$$[\delta x^{II}(T/2)\delta y^{III}(T/2)-\delta x^{III}(T/2)\delta y^{II}(T/2)]$$

so that the determinantal equation (25) takes the form

$$(26)\quad (1+s)^4(2,3)(1,4)+(1-s)^4(1,4)(2,3)+(1-s^2)^2$$

$$[(1,2)(3,4)-(1,3)(2,4)-(2,4)(1,3)+(3,4)(1,2)]=0$$

In this form we see that, in order for the characteristic equation to have a root equal to $+1$, it is necessary and sufficient that

(27) $$(2,3)(1,4) = 0.$$

Moreover, the matrix $R(T/2;0)$ has a determinant equal to $+1$. This determinant being obtained by making $s = 0$ in (26), we obtain the relation

(28) $$1 - (1,4)(2,3) = (1,2)(3,4) - (1,3)(2,4) - (2,4)(1,3) + (3,4)(1,2).$$

In view of (27) and (28), the characteristic equation (26) takes the simple form

$$(1-s^2)\left\{(1-s)^2(1,4)(2,3) + (1+s)^2[1-(1,4)(2,3)]\right\} = 0.$$

Consequently, the non-trivial characteristic roots are the solutions of the quadratic equation

$$1 + 2[1-2(1,4)(2,3)]s + s^2 = 0.$$

As in the general case, we write these two roots as

$$s_1 = je^{\Omega T} \qquad s_2 = je^{-\Omega T}$$

with $j = \pm 1$, so that

(29) $$1 + j\cosh\Omega T = 2(1,4)(2,3).$$

There ensues from it information concerning the stability of a symmetric periodic orbit; the conclusions are summarized in Table II.

TABLE II. VARIATIONAL STABILITY OF A SYMMETRIC PERIODIC ORBIT AS DEFINED BY ITS MONODROMY AT HALF THE PERIOD

	Ω	j	
$(1,4)(2,3) > 1$	real	+1	even instability
$(1,4)(2,3) = 1$	0	+1	indifferent case
$1/2 \leq (1,4)(2,3) < 1$	purely imaginary	+1	even stability
$0 < (1,4)(2,3) \leq 1/2$	purely imaginary	-1	odd stability
$(1,4)(2,3) = 0$	0	-1	indifferent case
$(1,4)(2,3) < 0$	real	-1	odd instability

6. RABE'S TROJAN ORBITS

In order to numerically calculate the orbits described in this section a double precision FORTRAN IV program for computing asymmetric periodic orbits in the plane restricted problem of three bodies was run on the IBM 7094 computer. In addition to calculating the basic dependent variables of an orbit with given initial conditions, the program provides for the calculation of the four independent solutions $\delta^{I}, \delta^{II}, \delta^{III}$, and δ^{IV} of the variational equations. If (for a given value of the period) the initial values are such that the orbit is truly periodic, the characteristic roots of the periodic orbit may be calculated. If the orbit is not as close to being periodic as may be desired

(i.e., if either $|x(T) - x(0)|$, $|y(T) - y(0)|$, $|p_x(T) - p_x(0)|$, or $|p_y(T) - p_y(0)|$ are larger than some given constant, such as for example 10^{-8} or 10^{-10}), then the program provides for changing the initial conditions slightly so that an orbit will be obtained which is closer to being periodic with the given period T. Of course, reasonable approximations to the initial conditions are necessary; it is not expected that the program will be asked to find a periodic orbit of given period starting with arbitrary guesses for initial conditions.

In order to explain this method of "improving orbits to make them truly periodic" some simplifying notation will be used. Let $\vec{x}(t;\vec{x}_0)$ represent the vector whose four components are $x(t;x_0,y_0,p_{x_0},p_{y_0})$, $y(t;x_0,y_0,p_{x_0},p_{y_0})$, $p_x(t;x_0,y_0,p_{x_0},p_{y_0})$ and $p_y(t;x_0,y_0,p_{x_0},p_{y_0})$, where these four variables represent the solution of equations (3) with the particular initial conditions $(x_0,y_0,p_{x_0},p_{y_0})$. Similarly, $\vec{\delta}^{I}(t;\vec{x}_0)$, $\vec{\delta}^{II}(t;\vec{x}_0)$, $\vec{\delta}^{III}(t,\vec{x}_0)$, $\vec{\delta}^{IV}(t;\vec{x}_0)$ represent the four linearly independent solutions of the variational equations. For the given period T, it is assumed that

$$\vec{x}(T;\vec{x}_0) \neq \vec{x}(0;\vec{x}_0)$$

although these vectors are not "too far" from being equal. Considering the first order variations, it is desired to consider solutions of the form

$$(30)\quad \vec{x}(t;\vec{x}_0 + \Delta\vec{x}_0) = \vec{x}(t;\vec{x}_0) + \alpha_1 \vec{\delta}^{I}(t;\vec{x}_0) + \alpha_2\vec{\delta}^{II}(t;\vec{x}_0) + \alpha_3\vec{\delta}^{III}(t;\vec{x}_0) + \alpha_4\vec{\delta}^{IV}(t;\vec{x}_0)$$

where $\Delta\vec{x}_0$ has components Δx_0, Δy_0, Δp_{x_0}, Δp_{y_0}.

Setting $t = 0$ and remembering the initial conditions (18), it is seen that one must have

$$\begin{aligned} \alpha_1 &= \Delta x_0 \\ \alpha_2 &= \Delta y_0 \\ \alpha_3 &= \Delta p_{x_0} \\ \alpha_4 &= \Delta p_{y_0} . \end{aligned} \tag{31}$$

If it is decided to use the given period T as a fixed quantity, then it will be desired to have the solution satisfy the conditions

$$\vec{x}(T;\vec{x}_0 + \Delta\vec{x}_0) = \vec{x}(0;\vec{x}_0 + \Delta\vec{x}_0) \equiv \vec{x}_0 + \Delta\vec{x}_0 .$$

Substituting this and equations (31) into equations (30) gives the results

$$\vec{x}_0 + \Delta\vec{x}_0 = \vec{x}(T;\vec{x}_0) + \Delta x_0 \vec{\delta}^{I}(T;\vec{x}_0) + \Delta v_0 \vec{\delta}^{II}(T;\vec{x}_0) + \Delta p_{x_0} \vec{\delta}^{III}(T;\vec{x}_0) + \Delta p_{y_0} \vec{\delta}^{IV}(T,\vec{x}_0) . \tag{32}$$

These linear equations are solved for the components of the unknown $\Delta\vec{x}_0$. If the new guesses $\vec{x}_0 + \Delta\vec{x}_0$ for initial conditions still do not produce a satisfactory periodic orbit, the process may be repeated. There is, however, a limit on how extremely close to a periodic orbit one can come by means of this iterative procedure. In general, we found that with 16-place arithmetic we could ensure that no component of $\vec{x}(T,\vec{x}_0)$ would differ from the corresponding component of $\vec{x}(0,\vec{x}_0)$ by more than 10^{-10}. However, if much better initial guesses are known and used, the equations (32) become extremely ill-conditioned.

If we had an exact periodic orbit, $\vec{x}_0$ would equal $\vec{x}(T;\vec{x}_0)$, and equations (32) would become precisely equations (20) with $s = +1$. That is, if we try to solve equations (32) under these conditions, it means that we are trying to find an eigenvector (corresponding to the known eigenvalue +1) for the matrix $R(T;0)$.

In describing his Trojan orbits, Rabe uses a jovicentric synodical coordinate system. The mass of the sun is taken as unity and the period of Jupiter is $2\pi\sqrt{1-\mu}$. If we use asterisks to represent variables used by Rabe, the transformation equations giving our units in terms of Rabe's are

$$\begin{aligned}
x &= 1 - \mu - x^* \\
y &= -y^* \\
\frac{dx}{dt} &= -\sqrt{1-\mu}\,\frac{dx^*}{dt^*} \\
\frac{dy}{dt} &= -\sqrt{1-\mu}\,\frac{dy^*}{dt^*} \qquad (33) \\
T &= \frac{1}{\sqrt{1-\mu}}\,T^* \\
C &= (1-\mu)C^*.
\end{aligned}$$

The orbital values given in Tables I and II of Rabe (1961) and Tables I and II of Rabe (1962) were transformed by means of these equations and by the equations

$$p_x = \frac{dx}{dt} - y$$

$$p_y = \frac{dy}{dt} + x$$

to give the initial values in Table III and the values of T and C given in Table IV.

In each case, the period T as given in Table IV was taken as a fixed parameter, and a more accurate periodic orbit was calculated as described in the preceding paragraphs. For these new orbits, the characteristic roots were calculated. In Table V are given the new initial values for the periodic orbits. Table VI lists the Jacobi constant as well as the trace of the monodromy matrix and the characteristic roots corresponding to the periodic orbit.

TABLE III. STARTING VALUES FOR PERIODIC ORBITS--{Rabe's results transformed to the units of this paper.

Rabe's Parameter d_o	x_0	y_0	$(p_x)_0$	$(p_y)_0$
1.0025	.500296124643	-.868190470	.864939351691	.498421029587
1.0050	.501546124643	-.870355530	.863857161536	.497798523295
1.0075	.502796124643	-.872520600	.862778849534	.497178495820
1.0100	.504046124643	-.874685658	.861704291736	.496561161060
1.0125	.505296124643	-.876850720	.860633590098	.495946217159
1.0150	.506546124643	-.879015780	.859566642668	.495333906001
1.0175	.507796124643	-.881180850	.858503443453	.494724073660
1.0200	.509046124643	-.883345912	.857444086401	.494116614187
1.0225	.510296124643	-.885510980	.856388739438	.493511365658
1.0250	.511546124643	-.887676040	.855340133254	.492908300088
1.0275	.512796124643	-.889841100	.854287389045	.492306843750
1.0300	.514046124643	-.892006166	.853245632595	.491707425438
1.0400	.519046124643	-.900666420	.849111400136	.489392102889
1.0500	.524046124643	-.909326674	.844638938108	.487334093530
1.0600	.529046124643	-.917986928	.840686966660	.484695798129

TABLE IV. PERIODS AND JACOBI CONSTANTS (using initial conds. in Table III)

Rabe's Parameter d_0	Period T	Jacobi Constant
1.0025	78.11824642410334	3.0000046197401
1.0050	78.21241134561734	3.0000184246002
1.0075	78.37095698004493	3.0000413337195
1.0100	78.59711486899776	3.0000732673572
1.0125	78.89553723182391	3.0001141457676
1.0150	79.27284722811138	3.0001638928252
1.0175	79.73765896723368	3.0002224296547
1.0200	80.30133787108203	3.0002896819513
1.0225	80.97856094134159	3.0003655816422
1.0250	81.79018812928275	3.0004502334866
1.0275	82.75360773001720	3.0005428247134
1.0300	83.91284074382773	3.0006443168953
1.0400	91.82099333962959	3.0011363774494
1.0500	117.7920128526034	3.0017245907688
1.0600	183.5145658929861	3.0024430527304

TABLE V. STARTING VALUES FOR PERIODIC ORBITS (PRESENT RESULTS)

Rabe's Parameter d_0 (approximate)	x_0	y_0	$(p_x)_0$	$(p_y)_0$
1.0025	.500294701009	-.868189394186	.864940407774	.498420857938
1.0050	.501546202315	-.870355763449	.863857082677	.497798432577
1.0075	.502795932420	-.872520424844	.862778961872	.497178566762
1.0100	.504046133194	-.874685581794	.861704299953	.496561215540
1.0125	.505296118194	-.876850529922	.860633607448	.495946396251
1.0150	.506546050763	-.879015868171	.859566684583	.495333815576
1.0175	.507796128489	-.881180842237	.858503483592	.494724052434
1.0200	.509045980033	-.883346025885	.857444171146	.494116467086
1.0225	.510296189709	-.885510985492	.856388586745	.493511547160
1.0250	.511545765989	-.887676141744	.855340268383	.492908273233
1.0275	.512795779277	-.889841341844	.854287586217	.492306477874
1.0300	.514045824807	-.892006380486	.853245786222	.491707120286
1.0400	.519045209040	-.900666970996	.849111898087	.489391219068
1.0500	.524045017193	-.909327242850	.844639557053	.487333137683
1.0600	.529053778602	-.917980837710	.840684004699	.484703533306

TABLE VI. JACOBI CONSTANTS, TRACES, AND CHARACTERISTIC ROOTS USING INITIAL CONDITIONS FROM TABLE V AND WITH SOME PERIODS AS IN TABLE IV

Rabe's Parameter d_0 (approximate)	Jacobi Constant C	Trace	Characteristic Roots
1.0025	3.0000046137301	.43885321	-.78057339 ± .62506414i
1.0050	3.0000184264690	.32970410	-.83514795 ± .55002537i
1.0075	3.0000413312188	.17882263	-.91055869 ± .41331374i
1.0100	3.0000732665714	.04109871	-.97945065 ± .20168399i
1.0125	3.0001141435380	.00832747	-.99583627 ± .09115993i
1.0150	3.0001638941519	.21110820	-.89444590 ± .44717617i
1.0175	3.0002224306764	.79229906	-.60385047 ± .79709762i
1.0200	3.0002896834712	1.81711725	-.09144138 ± .99581046i
1.0225	3.0003655812550	3.09222181	.54611090 ± .83771289i
1.0250	3.0004502346940	3.96383295	.98191648 ± .18931464i
1.0275	3.0005428266252	3.44565407	.72282704 ± .69102900i
1.0300	3.0006443182264	1.33281279	-.33359361 ± .94271698i
1.0400	3.0011363784399	.11223160	-.94388420 ± .33027658i
1.0500	3.0017245910778	1.17049332	-.41475334 ± .90993388i
1.0600	3.0024430503909	4.07526522	1.3145467 and .76071849

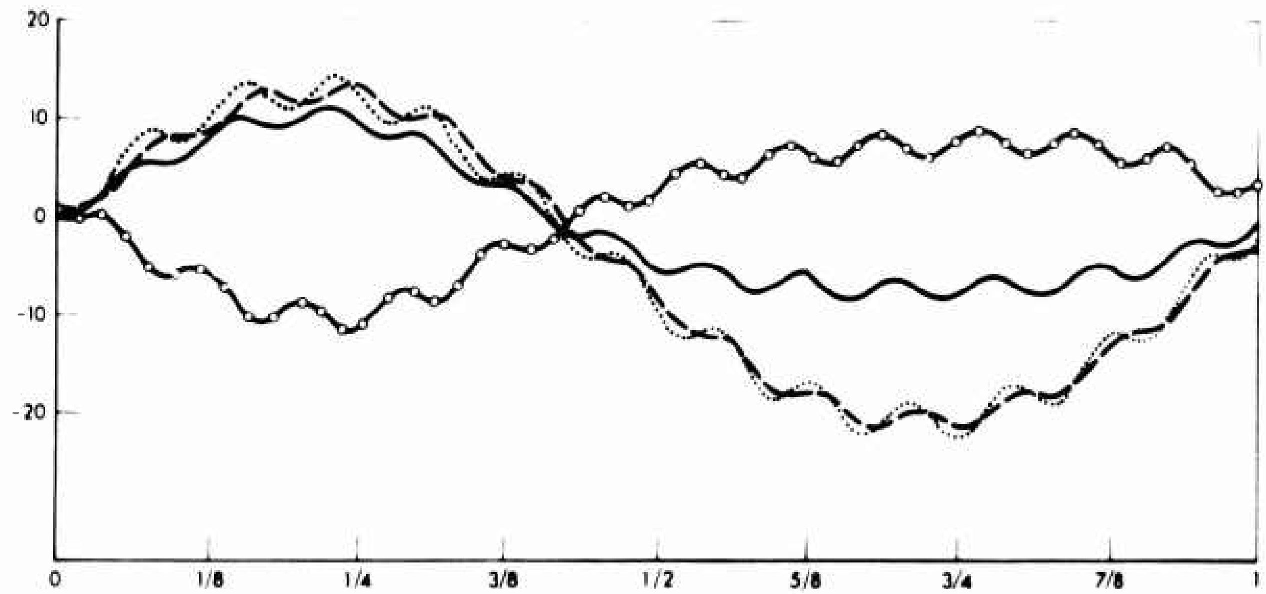

Fig. 1. Variation δx^{I} for Trojan Orbit (C = 3.0000603664498 and T = 78.50504948179581) versus time (the unit of time is the period T).

The orbits described by the parameters in Tables V and VI are practically the same as Rabe computed. The only reason we refined them slightly using our methods and 16-place arithmetic was to enable us to calculate the characteristic roots with good accuracy; this point will be discussed later. The fact that the entries in Table V are quite different from corresponding entries in Table III does not mean that the orbits themselves are very different; our iteration method has merely allowed for starting at a slightly different point on the orbit. The only real difference between the orbits is due to the fact that we took Rabe's 5-decimal place value for the period as being an exact constant, and found the more accurate orbit which has this exact period. Another indication that there is not too much difference between Rabe's original orbits and our modified ones is that the Jacobi constants as given in Table IV do not differ from the new Jacobi constants as given in Table VI by more than 6×10^{-9}.

In the way in which our computer program was used for the cases represented in the tables, the number of terms of the series was fixed at from 16 to 20. The step size was then determined so that none of the last three terms in any of the series would be more than 10^{-16}. At the end of the orbit, the number of terms taken in the series might in some cases be diminished in order to keep underflow from occurring. We considered an orbit to be periodic if no component of $\vec{x}(T,\vec{x}_0)$ differed from the corresponding component of $\vec{x}(0;\vec{x}_0)$ $\vec{x}_0$ by more than 10^{-10}. The Jacobi constant associated with the solution of the equations of motion remained the same to fifteen significant figu.es. The

Jacobi constants associated with the variational equations remained the same to eleven decimal places. The equations (19) were satisfied with residuals at most 10^{-9}.

In integrating the equations of motion it is feasible to use other methods of numerical integration, although probably more machine time would be involved. The power series method becomes relatively more advantageous when integrating the variational equations. The solutions of these equations are highly oscillatory in nature, and a high order numerical integration method is essential if an extremely small step size and multiple precision in adding on the increments is not to be required. Figure 1 shows a sketch of the four components of one variational equation solution. The points showing on one curve are those which were required to be calculated by the power series method using 16 terms of the series. (In actually drawing the curves, intermediate points were also calculated so as to present the true shape for illustration.) Since the basic orbital variables occur in the variational equations, the step size required for integration of the variational equations is then the step size for the whole problem when the integrations are done together in an efficient manner.

In many types of problems, loosely approximate solutions of variational equations may be sufficient, but in this case it is imperative that the values of the four variational solutions be known right at the end of the period. Because of the rapid changes in the four solutions, the trace of the matrix of the solutions also changes rapidly, and the calculated characteristic roots would not be correct if the orbit were not very close to being periodic and if the variational equations were not solved very accurately. In order to be

absolutely sure that the accuracies described in the previous paragraphs were indeed good enough to permit determination of the characteristic roots accurately, we chose initial conditions which were approximations to orbital values at various parts of the orbit. When truly periodic orbits were obtained using these initial conditions as first guesses, it was found that the calculated characteristic roots were the same to eight decimal places as they were when a start was made at a different point on the orbit.

In accordance with the criteria given in Table I, the results in Table VI show that all of Rabe's orbits are stable except the last one. The stable orbits are of oval shape. The one unstable orbit is of horseshoe shape. It is evident from looking at the columns listing the trace and the characteristic roots that as the periods of the various orbits increase, the characteristic roots travel around on the unit circle. There will be points of "indifferent stability" as indicated in Table I when the path of the roots actually hits the point -1 or $+1$.

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APPENDIX: COMPUTER PROGRAM WRITE-UP

Program 36 - "Non-symmetric orbits with variational solutions"

1. General Information

A. Purpose

The purpose of this program is to calculate accurate periodic orbits in the plane restricted problem of three bodies. Four independent solutions of the variational equations may also be computed so that the characteristic roots associated with the orbit may be obtained.

B. Restrictions

1. This is a FORTRAN IV program which has been run on an IBM 7094 under the IBSYS system in which tape 5 is the input tape, tape 6 is the output tape for printing, and tape 14 is the output tape for punching.

2. All input variables are either fixed point numbers or double precision floating point numbers.

3. For a desired orbit the exact period must be given. Approximate initial values of x, y, p_x, and p_y are also required; arbitrary starting values will in general not be good enough.

4. If one asks for extreme accuracy in finding a periodic orbit, the program will obtain orbits which are closer and closer to being periodic for a while, and then next orbit

or orbits will not be so good again. The reason for this is given on page 34. Thus if guesses for initial conditions are originally extremely good, the next supposedly improved guesses will not be as good. In this regard, the values of the FORTRAN variable C7 has the effect of determining whether or not these factors are a problem in a particular case.

C. Method

1. Mathematical Method

This is described in the main body of the document. However, as stated there, some of the actual equations used in the computer program are slightly different from those given in the body of the document. There are 4 dependent variables in equations (3); in the 4 independent solutions of the variational equations (9) with initial conditions (18), 16 more dependent variables occur. In using the recurrent power series method for solving the 20 equations, we here introduce 13 auxiliary variables as follows (instead of 17 as described earlier):

$$a \equiv (x+\mu)^2 \qquad B \equiv 1 - \frac{3(x+\mu-1)^2}{(x+\mu-1)^2+y^2}$$

$$d \equiv (x+\mu)y$$

$$w \equiv (x+\mu)^2 + y^2 \qquad C \equiv \frac{(x+\mu)y}{(x+\mu)^2+y^2}$$

$$g \equiv \frac{1}{(x+\mu)^2+y^2} \qquad D \equiv \frac{(x+\mu-1)y}{(x+\mu-1)^2+y^2}$$

$$h \equiv \frac{1}{(x+\mu-1)^2+y^2} \qquad R \equiv (1-\mu)[(x+\mu)^2+y^2]^{-3/2}$$

$$A \equiv 1 - \frac{3(x+\mu)^2}{(x+\mu)^2+y^2} \qquad S \equiv [(x+\mu-1)^2+y^2]^{-3/2} \tag{34}$$

$$L \equiv \frac{(1-\mu)}{[(x+\mu)^2+y^2]^{3/2}}\left[1-\frac{3(x+\mu)^2}{(x+\mu)^2+y^2}\right] + \frac{\mu}{[(x+\mu-1)^2+y^2]^{3/2}}\left[1-\frac{3(x+\mu-1)^2}{(x+\mu-1)^2+y^2}\right]$$

$$N \equiv \frac{3(1-\mu)(x+\mu)y}{[(x+\mu)^2+y^2]^{5/2}} + \frac{3\mu(x+\mu-1)y}{[(x+\mu-1)^2+y^2]^{5/2}} \, .$$

If these definitions are written in the form

$$a = x^2 + 2\mu x + \mu^2$$

$$d = xy + \mu y$$

$$w = a + y^2$$

$$g = 1/w$$

$$h = \frac{1}{w - 2x + (1 - 2\mu)}$$

$$A = 1 - 3ga$$

(35) $$B = 1 - 3ha + 6hx - 3(1 - 2\mu)h$$

$$C = gd$$

$$D = hd - hy$$

$$R = (1 - \mu)g^{3/2}$$

$$S = \mu h^{3/2}$$

$$L = RA + SB$$

$$N = 3RC + 3SD,$$

it is seen that in most of the definitions the right-hand side only consists of linear combinations of products or quotients of at most two of the other variables. In the case of R and S, the equation may be differentiated so that

(36) $$2g\dot{R} = 3R\dot{g}$$ $$2h\dot{S} = 3S\dot{h}$$

which will also be satisfactory for the recurrent power series method.

In all we have 33 variables to consider, (20 desired dependent variables as well as the 13 auxiliary variables discussed above). Each of these variables is considered as a power series. For example,

$$A = \sum_{n=1}^{\infty} A_n (\Delta t)^{n-1}$$

and

$$x = \sum_{n=1}^{\infty} x_n (\Delta t)^{n-1} .$$

If convenient, the coefficients of the power series use the same symbol as the dependent variable itself but are subscripted with an n. The exceptions to this show up in the definitions

$$p_x = \sum_{n=1}^{\infty} p_n (\Delta t)^{n-1} \qquad p_y = \sum_{n=1}^{\infty} q_n (\Delta t)^{n-1}$$

$$\delta x^{(i)} = \sum_{n=1}^{\infty} k_n^{(i)} (\Delta t)^{n-1}, \qquad i = I, II, III, IV,$$

$$\delta y^{(i)} = \sum_{n=1}^{\infty} \ell_n^{(i)} (\Delta t)^{n-1}, \qquad i = I, II, III, IV,$$

$$\delta p_x^{(i)} = \sum_{n=1}^{\infty} u_n^{(i)} (\Delta t)^{n-1}, \qquad i = \mathrm{I,II,II,IV},$$

$$\delta p_y^{(i)} = \sum_{n=1}^{\infty} v_n^{(i)} (\Delta t)^{n-1}, \qquad i = \mathrm{I,II,III,IV}.$$

When these power series are substituted into equations (35) [or (36)] and into the equations of motion (3) and in the variational equations (9), and the corresponding powers of Δt are equated, the following recursion formulas are obtained:

(37.1) $$n x_{n+1} = p_n + y_n$$

(37.2) $$n y_{n+1} = q_n - x_n$$

(37.3) $$a_{n+1} = 2\mu x_{n+1} + \sum_{j=1}^{n+1} x_j x_{n+2-j}$$

(37.4) $$w_{n+1} = a_{n+1} + \sum_{j=1}^{n+1} y_j y_{n+2-j}$$

(37.5) $$g_{n+1} = -\frac{1}{w_1} \sum_{j=2}^{n+1} w_j g_{n+2-j}$$

(37.6) $$h_{n+1} = h_1 \sum_{j=2}^{n+1} (2x_j h_{n+2-j} - w_j h_{n+2-j})$$

(37.7) $$R_{n+1} = \frac{1}{2g_1}\left\{3g_{n+1}R_1 - \frac{1}{n}\sum_{j=1}^{n-1} j[2R_{j+1}g_{n+1-j} - 3g_{j+1}R_{n+1-j}]\right\}$$

(37.8) $$S_{n+1} = \frac{1}{2h_1}\left\{3h_{n+1}S_1 - \frac{1}{n}\sum_{j=1}^{n-1} j[2S_{j+1}h_{n+1-j} - 3h_{j+1}S_{n+1-j}]\right\}$$

(37.9) $$d_{n+1} = \mu y_{n+1} + \sum_{j=1}^{n+1} x_j y_{n+2-j}$$

(37.10) $$A_{n+1} = -3\sum_{j=1}^{n+1} g_j a_{n+2-j}$$

(37.11) $$B_{n+1} = -3(1-2\mu)h_{n+1} - 3\sum_{j=1}^{n+1} (h_j a_{n+2-j} - 2h_j x_{n+2-j})$$

(37.12) $$C_{n+1} = \sum_{j=1}^{n+1} g_j d_{n+2-j}$$

(37.13) $$D_{n+1} = \sum_{j=1}^{n+1} (h_j d_{n+2-j} - h_j y_{n+2-j})$$

(37.14) $$L_{n+1} = \sum_{j=1}^{n+1} (R_j A_{n+2-j} + S_j B_{n+2-j})$$

(37.15) $$N_{n+1} = 3\sum_{j=1}^{n+1} (R_j C_{n+2-j} + S_j D_{n+2-j})$$

(37.16) $$np_{n+1} = q_n - \mu R_n + (1-\mu)S_n - \sum_{j=1}^{n} (R_j + S_j)x_{n+1-j}$$

(37.17) $$nq_{n+1} = -p_n - \sum_{j=1}^{n} (R_j + S_j)y_{n+1-j}$$

(37.18) $$nk_{n+1}^{(i)} = \ell_n^{(i)} + u_n^{(i)}, \qquad i = I,II,III,IV$$

(37.19) $$n\ell_{n+1}^{(i)} = -k_n^{(i)} + v_n^{(i)}, \qquad i = I,II,III,IV$$

(37.20) $$nu_{n+1}^{(i)} = v_n^{(i)} - \sum_{j=1}^{n} L_j k_{n+1-j}^{(i)} + \sum_{j=1}^{n} N_j \ell_{n+1-j}^{(i)}, \qquad i = I,II,III,IV$$

(37.21) $$nv_{n+1}^{(i)} = -u_n^{(i)} + \sum_{j=1}^{n} (R_j+S_j)\ell_{n+1-j}^{(i)} + \sum_{j=1}^{n} L_j \ell_{n+1-j}^{(i)} + \sum_{j=1}^{n} N_j k_{n+1-j}^{(i)}, \qquad i = I,II,III,IV.$$

The known initial conditions are the quantities x [illegible] v_1, p_1, $q_1, k_1^{(i)}, \ell_1^{(i)}, u_1^{(i)}, v_1^{(i)}$, $i = I,II,III,IV$. The [illegible] der,

(38.1) $$a_1 = (x_1 + \mu)^2$$

(38.2) $$w_1 = a_1 + y_1^2$$

(38.3) $$g_1 = 1/w_1$$

(38.4) $$h_1 = \frac{1}{w_1 - 2x_1 + 1 - 2\mu}$$

(38.5) $$R_1 = (1 - \mu)g_1 \sqrt{g_1}$$

(38.6) $S_1 = \mu h_1 \sqrt{h_1}$

(38.7) $d_1 = (x_1 + \mu) y_1$

(38.8) $A_1 = 1 - 3 g_1 a_1$

(38.9) $B_1 = 1 - 3 h_1 (a_1 - 2x_1 + 1 - 2\mu)$

(38.10) $C_1 = g_1 d_1$

(38.11) $D_1 = h_1 (d_1 - y_1)$

(38.12) $L_1 = R_1 A_1 + S_1 B_1$

(38.13) $N_1 = 3(R_1 C_1 + S_1 D_1)$

Now equations (37) may be used in the order listed to obtain the second coefficients in each series, etc. When the program is being run in the mode in which only the equations of motion (and not the variational equations) are being solved, equations (37.9) through (37.15) and equations (37.18) through (37.21) as well as equations (38.7) through (38.13) are not used in the recurrence process.

For the mathematical method of improving the "periodic" orbit and for obtaining the characteristic roots associated with the periodic orbit, see the explanation in the main body of the document.

2. Coding Method

IBM 7094 FORTRAN IV. This program and its subroutines make maximum use of COMMON storage. We have desired to run many similar types of problems without changing much of the program. For this reason there are many COMMON variables available which were not actually used in this version of the program. These show up as being undefined in the listing of what each COMMON variable represents.

COMMON Variables

Note: A(1,I) thru A(9,I) are usually as defined below. However, at the end of the orbit under the control of subroutine ENDOR, they are used as matrix elements.

A(1,I) thru A(12,I)	$x_i, y_i, p_i, q_i,$	$k_i^I, \ell_i^I, u_i^I, v_i^I,$	$k_i^{II}, \ell_i^{II}, u_i^{II}, v_i^{II}$
A(13,I) thru A(24,I)	$k_i^{III}, \ell_i^{III}, u_i^{III}, v_i^{III},$	$k_i^{IV}, \ell_i^{IV}, u_i^{IV}, v_i^{IV},$	a_i, w_i, g_i, h_i
A(25,I) thru A(33,I)	$R_i, S_i, d_i, A_i,$	B_i, C_i, D_i, L_i	N_i
A(34,I) thru A(49,I)			
A(50,I)	$R_i + S_i$		
B(1) thru B(12)	$x, y, p_x, p_y,$	$\delta x^I, \delta v^I, \delta p_x^I, \delta p_y^I,$	$\delta x^{II}, \delta y^{II}, \delta p_x^{II}, \delta p_y^{II}$
B(13) thru B(20)	$\delta x^{III}, \delta y^{III}, \delta p_x^{III}, \delta p_y^{III},$	$\delta x^{IV}, \delta y^{IV}, \delta p_x^{IV}, \delta p_y^{IV}$	
B(21) thru B(50)			

C(1) thru C(5)	$x(0), y(0), p_x(0), p_y(0)$, Jacobi Constant at $t=0$.
C(6) thru C(50)	
E(1)	Trace of monodromy matrix
E(2)	Real part of 1st characteristic root
E(3)	Imaginary part of 1st characteristic root
E(4)	Real part of 2nd characteristic root
E(5) thru E(50)	
F(I)	
GI(I)	i
GII(I)	$1/i$
S(I)	$(\Delta t)^{i-1}$
x(I)	
y(I)	
z(I)	Erasable storage. (However, at the end of an orbit, z(I) <u>is</u> carried to subroutine ORBIT through subroutines MATR, GUES, and ENDOR)
C1,C2	
C3	Parameter used to help determine Δt. (Making C3 smaller improves accuracy attainable). C3 is usually chosen between .1 and 1. and mostly about .25.

C4	
C5	Period of the orbit
C6	
C7	Accuracy required in calling orbit "periodic". (This is the allowable difference between any of the 4 initial values of the orbital variables and the corresponding value at the end of a period.)
C8 thru C10	
D	Max Δt which subroutine POWER thinks should be used
DT	Δt which is actually used
DTMAX	Maximum Δt ever to be used
DSMAX	
E1 thru E10	
G	
GMU	μ
GMU1	$\mu - 1$
GMUC	$1 - 2\mu$
H	

I,II,J,K,K1 thru K10	Erasable
L	M - 1
L1	N + 1
L2	N + 2
L3 thru L8	
L9	Usually 0. If 1, subroutine GUES prints dump of matrices and solutions.
L10	Max. number of times subroutine ORBIT may be called.
M	Number of terms used in Taylor series.
M1	Case number. Later this is used by subroutine ORBIT to tell whether it is through (M1 = 1) or not (M1 = 0).
M2	If M2 = 4, only the 4 equations of motion are solved. If M2 = 20, the variational equations are solved also.
M3	Number of integration steps per printing interval.
M4 thru M6	
M7	If M7 = $\left\{\begin{array}{l}\text{1, there is to be no}\\ \text{2, there is to be}\end{array}\right\}$ intermediate printing of variational quantities.
M8	
M9	Max. number of times subroutine POWER may be called
M10	

N	Running variable (N = 1,M)
N1	If $N1 = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$, the program prints $\begin{Bmatrix} \text{only 4} \\ \text{all 20} \end{Bmatrix}$ variables at intermediate points on orbit.
N2	If $N2 = \begin{Bmatrix} \text{1, only 4} \\ \text{2, all 20} \end{Bmatrix}$ variables are calculated.
N3	A counter to compare with M3
N4	If $N4 = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$, subroutine PRNT $\begin{Bmatrix} \text{is not} \\ \text{is} \end{Bmatrix}$ called by subroutine ORBIT.
N5	Ordinarily N5 = 0. Subroutine SETDT sets N5 = 1 when it has found a suitable Δt which will just end an orbit.
N6	
N7	If $N7 = \begin{Bmatrix} \text{1; there is now no} \\ \text{2; there is now} \end{Bmatrix}$ intermediate printing of variational quantities.
N8	
N9	Max. number of times POWER may <u>still</u> be called.
N10	
P	$\|\|\Delta\vec{x}_0\|\|$
Q,R	
T	independent variable (time)
U,V,W	
x1 thru x10	

y1 thru y10

z1 thru z10 Erasable storage

II. Running Procedure

A. There are many subroutines used by this program. In the order that they are called, the basic four are:

RE36D -- This routine reads the necessary input data for the problem and prints it back out again.

SETUP -- This routine initializes the necessary quantities to start an orbit. It prints headings and calls PRNT to print initial values in the table.

ORBIT -- This routine runs a complete orbit. In doing so it uses the subroutines:

POWER -- This routine calculates the series coefficients and computes a likely size for $D \equiv \Delta t_{max}$.

SETDT -- This routine decides just what Δt to actually use. When needed it calls subroutine.

TAYL -- This subroutine calculates the dependent variables for a given Δt by the Taylor series.

SETPR -- This routine decides whether any printing is to be done at the particular time in accordance with input parameter information. If so, it tells the program ORBIT to call.

PRNT -- This routine prints the orbital parameters for the given value of t.

ENDOR -- At the end of the orbit, this routine calculates and prints residuals in checking equations (19). It calculates and prints the trace and the characteristic roots. Then it calls GUES -- This routine sets up equations (32). In order to solve them it calls subroutine MATR. The suggested corrections Δx_0, Δy_0, Δp_{x_0}, Δp_{y_0} are printed, and the new values of C(I) I = 1,4 are set up.

The subroutine ORBIT then decides if it has obtained a good enough periodic orbit. It transfers this information to the main program by means of parameter M1.

FANPR -- This subroutine prints all necessary information (which might be saved for long periods of time) about the true periodic orbit.

The main program also punches (actually writes output tape 14) two cards which give $\mu, x(0), y(0)$, Period, $p_x(0), p_y(0)$ in the format required as input to the program.

B. Data input cards.

There are 5 input cards required for each case. Any number of cases may be run one right after the other. The cards are as follows: (When a quantity is given as "arbitrary" it means that this

input quantity will never actually be used in this particular version of the program. Of course, if one wishes to always use only this exact version of the program, subroutine RE36L could be easily changed so as to read only the quantities actually needed.)

Card 1: Format (3D24.16) μ x(0) y(0)

Card 2: Format (3D24.16) Period $p_x(0)$ $p_y(0)$

Card 3: Format (14I5) M = no. of terms in Taylor series

M2 = 4 or 20 (20 means variational equations also computed)

M3 = no. of integration steps per printing interval

M4 = arbitrary

M5 = arbitrary

M6 = arbitrary

M7 = 1 or 2 (2 means there is to be intermediate printing of variational quantities)

M8 = arbitrary

M9 = max. no. of times subroutine POWER may be called (i.e., the max. no. of integration steps you wish to allow without giving up on ever getting to the end of the period).

M10 = arbitrary

L10 = Max. no. of times subroutine ORBIT may be called (i.e., how many complete orbits are you going to allow; if L10 = 0 it will run one orbit).

L9 = 0 or 1 (1 means GUES prints a dump of matrices and solutions)

L8 = arbitrary

L7 = arbitrary

Card 4: Format (7D10.2)

C1 = arbitrary

C2 = arbitrary

C3 = parameter used to help determine Δt. (Making C3 smaller tends to cut Δt). C3 is usually chosen between .1 and 1. and mostly about .25.

C4 = arbitrary

C6 = arbitrary

C7 = accuracy required in called orbit "periodic". This is the allowable difference between any of the 4 initial values of the dependent variables and corresponding values at the end of a period.

C8 = arbitrary

Card 5: Format (4D10.2)

C9 = arbitrary

C10 = arbitrary

DTMAX = max. value of Δt you would ever want it to use

DSMAX = arbitrary

C. Output

Printed output depends on input parameters M3, M7, and L9.

Subroutine RE36D always prints the input data it has read. On

a new page will then come a table giving various orbital profiles. (If M3 is very large, one may only get initial values and values at the end of a period). The table will generally be in the format

T	x	y	$\dot{x}$	$\dot{y}$	C
	δx^{I}	δy^{I}	δp_x^{I}	δp_y^{I}	C^{I}
	δx^{II}	δy^{II}	δp_x^{II}	δp_y^{II}	C^{II}
	δx^{III}	δy^{III}	δp_x^{III}	δp_y^{III}	C^{III}
	δx^{IV}	δy^{IV}	δp_x^{IV}	δp_y^{IV}	C^{IV}
T	x	y	$\dot{x}$	$\dot{y}$	C
	δx^{I}	δy^{I}	δp_x^{I}	δp_y^{I}	C^{I}
	δx^{II}	δy^{II}	δp_x^{II}	δp_y^{II}	C^{II}
	δx^{III}	δy^{III}	δp_x^{III}	δp_y^{III}	C^{III}
	δx^{IV}	δy^{IV}	δp_x^{IV}	δp_y^{IV}	C^{IV}

etc.

Of course if M7 = 1, there will be no intermediate printing of the variational solutions.

At the end of an orbit which the program thinks is truly periodic (or even if it has been unsuccessful in really finding a good one), the six variational equation checks are printed.

(These are the residues obtained when checking in equations (19), and they should all be very close to zero).

Next are printed two characteristic roots and the truce of the monodromy matrix. Then are printed suggested changes in initial values as obtained as a solution of equations (32). Then the new initial guesses are printed. If the program has been successful in finding an accurate periodic orbit, one will usually not be interested in these "suggested changes in initial values" or in the "new guesses".

Finally the program prints a listing of important orbital parameters suitable for saving as a page in a book of orbits. If L9 = 1, the print out will be interspersed with matrix print outs by subroutine GUES. These are not labelled, and the easiest way to see what they mean is to read the program listing.

Punched output always consists of two cards in format (3D24.16) giving for the latest orbit μ, x(0), y(0), Period, $p_x(0)$, $p_y(0)$.

D. Results

An example giving the print out for a particular trojan orbit is given in the following pages.

```
        PRG36                                                                      05/07/65
     EXTERNAL FORMULA NUMBER    -    SOURCE STATEMENT     -   INTERNAL FORMULA NUMBER(S)

C      PROGRAM 36 NON-SYMMETRIC ORBITS WITH VARIATIONS
       DOUBLE PRECISION A, B, C, C1, C2, C3, C4, C5, C6, C7, C8, C9, C10,
      1D, DT, DTMAX, DSMAX, E, E1, E2, E3, E4, E5, E6, E7, E8, E9, E10,
      2F, G, GI, GII, GMU, GMU1, GMUC, H, P, Q, R, S, T, U, V, W, X1, X2,
      3X3, X4, X5, X6, X7, X8, X9, X10, Y1, Y2, Y3, Y4, Y5, Y6, Y7, Y8,
      4Y9, Y10, Z, Z1, Z2, Z3, Z4, Z5, Z6, Z7, Z8, Z9, Z10
       DIMENSION A(50,50), B(50), C(50), E(50), F(50), GI(50), GII(50),
      1X(2000), Y(2000), Z(100), S(50)
       COMMON/SPR/X, Y
       COMMON/DPR/A, B, C, C1, C2, C3, C4, C5, C6, C7, C8, C9, C10, D,
      1DT, DTMAX, DSMAX, E, E1, E2, E3, E4, E5, E6, E7, E8, E9, E10, F,
      2G, GI, GII, GMU, GMU1, GMUC, H, P, Q, R, S, T, U, V, W, X1, X2,
      3X3, X4, X5, X6, X7, X8, X9, X10, Y1, Y2, Y3, Y4, Y5, Y6, Y7, Y8,
      4Y9, Y10, Z, Z1, Z2, Z3, Z4, Z5, Z6, Z7, Z8, Z9, Z10
       COMMON/INTS/I, II, J, K, K1, K2, K3, K4, K5, K6, K7, K8, K9, K10,
      1L, L1, L2, L3, L4, L5, L6, L7, L8, L9, L10, M, M1, M2, M3, M4. M5,
      2M6, M7, M8, M9, M10, N, N1, N2, N3, N4, N5, N6, N7, N8, N9, N10
     1 FORMAT (1H0 5D23.15)
     2 FORMAT (12H0NEW GUESSES )
    11 FORMAT (3D24.16)
    12 FORMAT (1H1)
     3 CALL RE36D                                                          .1
       P=1.D+30                                                            .2
     4 CALL SETUP                                                          .3
       CALL ORBIT                                                          .4
       IF (M1) 4,4,20                                                      .5
    20 DO 21 I=1,4                                                         .6
    21 Z(I)=C(I)+Z(I)                                                      .7    .8
       WRITE (6,2)                                                         .9    .10
       WRITE (6,1)      (Z(I),I=1,4)                                       .11   .12   .13   .14   .15
       CALL FANPR                                                          .16
                                                                           .17
       WRITE (6,12)                                                        .18   .19
       WRITE (14,11) GMU,C(1),C(2),C5,C(3),C(4)                            .20   .21   .22
       GO TO 3                                                             .23
       END                                                                 .24

BEGIN ASSEMBLY    14.020
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```
      SUBROUTINE ENDOR
      DOUBLE PRECISION A, B, C, C1, C2, C3, C4, C5, C6, C7, C8, C9, C10,
     1D, DT, DTMAX, DSMAX, E, E1, E2, E3, E4, E5, E6, E7, E8, E9, E10,
     2F, G, GI, GII, GMU, GMU1, GMUC, H, P, Q, R, S, T, U, V, W, X1, X2,
     3X3, X4, X5, X6, X7, X8, X9, X10, Y1, Y2, Y3, Y4, Y5, Y6, Y7, Y8,
     4Y9, Y10, Z, Z1, Z2, Z3, Z4, Z5, Z6, Z7, Z8, Z9, Z10
      DIMENSION A(50,50), B(50), C(50), E(50), F(50), GI(50), GII(50),
     1X(2000), Y(2000), Z(100), S(50)
      COMMON/SPR/X, Y
      COMMON/DPR/A, B, C, C1, C2, C3, C4, C5, C6, C7, C8, C9, C10, D,
     1DT, DTMAX, DSMAX, E, E1, E2, E3, E4, E5, E6, E7, E8, E9, E10, F,
     2G, GI, GII, GMU, GMU1, GMUC, H, P, Q, R, S, T, U, V, W, X1, X2,
     3X3, X4, X5, X6, X7, X8, X9, X10, Y1, Y2, Y3, Y4, Y5, Y6, Y7, Y8,
     4Y9, Y10, Z, Z1, Z2, Z3, Z4, Z5, Z6, Z7, Z8, Z9, Z10
      COMMON/INTS/I, II, J, K, K1, K2, K3, K4, K5, K6, K7, K8, K9, K10,
     1L, L1, L2, L3, L4, L5, L6, L7, L8, L9, L10, M, M1, M2, M3, M4, M5,
     2M6, M7, M8, M9, M10, N, N1, N2, N3, N4, N5, N6, N7, N8, N9, N10
    1 FORMAT(33H0SIX VARIATIONAL EQUATION CHECKS 6D12.5)
    2 FORMAT(1H0)
    3 FORMAT(112H0CHARACTERISTIC EXPONENTS
     1                                                   TRACE
    4 FORMAT(1H02D21.14,D28.14,D21.14,D28.14)
      N7 = 2                                                    ,1
      CALL PRNT                                                 ,2
      DO 8 I=1, 3                                               ,3
    8 Z(I) = B(5)*B(I+13)+B(9)*B(I+17)-B(13)*B(I+5)-B(17)*B(I+9) ,4   ,5
      Z(2) = Z(2)-1.                                            ,6
      DO 9 I=4, 5                                               ,7
    9 Z(I) = B(6)*B(I+11)+B(10)*B(I+15)-B(14)*B(I+3)-B(18)*B(I+7) ,8  ,9
      Z(5) = Z(5)-1.                                            ,10
      Z(6) = B(7)*B(16)+B(11)*B(20)-B(15)*B(8)-B(19)*B(12)     ,11
      WRITE (6,2)                                               ,12  ,13
      WRITE (6,1)(Z(I),I=1,6)                                   ,14  ,15  ,16  ,17  ,18
      Z1 = B(5)+B(10)+B(15)+B(20)                               ,19
      Z2 = .25*Z1-1.                                            ,20
      WRITE (6,3)                                               ,21  ,22
      Z4 = Z2+Z2+1.                                             ,23
      IF (Z1) 12, 30, 10                                        ,24
   10 IF (Z2) 20, 40, 12                                        ,25
   12 Z3 = DSQRT(Z1*Z2)                                         ,26
      Z6 = Z4-Z3                                                ,27
      Z4 = Z4+Z3                                                ,28
      GO TO 45                                                  ,29
   20 Z3 = -Z1*Z2                                               ,30
      Z5 = DSQRT(Z3)                                            ,31
      Z6 = Z4                                                   ,32
      Z7 = -Z5                                                  ,33
      GO TO 47                                                  ,34
   30 Z4 = -1.                                                  ,35
      Z6 = -1.                                                  ,36
      GO TO 45                                                  ,37
   40 Z4 = 1.                                                   ,38
      Z6 = 1.                                                   ,39
   45 Z5 = 0.                                                   ,40
      Z7 = 0.                                                   ,41
   47 E(1)=Z1                                                   ,42
      E(2)=Z4                                                   ,43
      E(3)=Z5                                                   ,44
      E(4)=Z6                                                   ,45
      WRITE (6,4) Z4, Z5, Z6, Z7, Z1                            ,46  ,47  ,48
      CALL GUES                                                 ,49
      N7 = M7                                                   ,50
      RETURN                                                    ,51
      END                                                       ,52

BEGIN ASSEMBLY    05.614
```

```
      SUBROUTINE FANPR
      DOUBLE PRECISION A, B, C, C1, C2, C3, C4, C5, C6, C7, C8, C9, C10,
     1D, DT, DTMAX, DSMAX, E, E1, E2, E3, E4, E5, E6, E7, E8, E9, E10,
     2F, G, GI, GII, GMU, GMU1, GMUC, H, P, Q, R, S, T, U, V, W, X1, X2,
     3X3, X4, X5, X6, X7, X8, X9, X10, Y1, Y2, Y3, Y4, Y5, Y6, Y7, Y8,
     4Y9, Y10, Z, Z1, Z2, Z3, Z4, Z5, Z6, Z7, Z8, Z9, Z10
      DIMENSION A(50,50), B(50), C(50), E(50), F(50), GI(50), GII(50),
     1X(2000), Y(2000), Z(100), S(50)
      COMMON/SPR/X, Y
      COMMON/DPR/A, B, C, C1, C2, C3, C4, C5, C6, C7, C8, C9, C10, D,
     1DT, DTMAX, DSMAX, E, E1, E2, E3, E4, E5, E6, E7, E8, E9, E10, F,
     2G, GI, GII, GMU, GMU1, GMUC, H, P, Q, R, S, T, U, V, W, X1, X2,
     3X3, X4, X5, X6, X7, X8, X9, X10, Y1, Y2, Y3, Y4, Y5, Y6, Y7, Y8,
     4Y9, Y10, Z, Z1, Z2, Z3, Z4, Z5, Z6, Z7, Z8, Z9, Z10
      COMMON/INTS/I, II, J, K, K1, K2, K3, K4, K5, K6, K7, K8, K9, K10,
     1L, L1, L2, L3, L4, L5, L6, L7, L8, L9, L10, M, M1, M2, M3, M4, M5,
     2M6, M7, M8, M9, M10, N, N1, N2, N3, N4, N5, N6, N7, N8, N9, N10
    1 FORMAT(64H0                                                  SUN
     1 JUPITER )
    2 FORMAT(63H0                                                  EAR
     1TH MOON )
    3 FORMAT(65H0                                                  EQU
     1AL MASSES )
    4 FORMAT(53H0                                             RM = D24
     1.16)
    5 FORMAT(53H0                             JACOBI CONSTANT = D24
     1.16)
    6 FORMAT(53H0                                        PERIOD = D24
     1.16)
    7 FORMAT(56H0ONE OF THE CONJUGATE COMPLEX CHARACTERISTIC ROOTS IS
     12D24.16)
    8 FORMAT(56H0                  THE TWO REAL CHARACTERISTIC ROOTS ARE
     12D24.16)
    9 FORMAT(56H0                                      THE TRACE WAS
     1D24.16)
   10 FORMAT(109H0                          X                          Y
     1                 P SUB X                   P SUB Y )
   11 FORMAT(17H0INITIAL VALUES  4D25.16)
   12 FORMAT(17H0  FINAL VALUES  4D25.16)
   13 FORMAT(1H0)
   14 FORMAT(36H0INITIAL DX/DT AND DY/DT VALUES WERE 2D24.16)
   15 FORMAT(75H1                                        PLANE RESTRICTED
     1 THREE BODY PROBLEM)
      WRITE(6,15)                                                    .1    .2
      IF(GMU-1.D-03)20,20,25                                         .3
   20 WRITE(6,1)                                                     .4    .5
      GO TO 40                                                       .6
   25 IF(GMU-.01250D)28,28,30                                        .7
   28 WRITE(6,2)                                                     .8    .9
      GO TO 40                                                       .10
   30 IF(GMU-.5D0)40,35,40                                           .11
   35 WRITE(6,3)                                                     .12   .13
   40 WRITE(6,4)GMU                                                  .14   .15   .16




      WRITE(6,5)C(5)                                                 .17   .18   .19
      WRITE(6,6)C5                                                   .20   .21   .22
      WRITE(6,13)                                                    .23   .24
      WRITE(6,10)                                                    .25   .26
      WRITE(6,11)(C(I),I=1,4)                                        .27   .28   .29   .30   .31
      WRITE(6,12)(B(I),I=1,4)                                        .32   .33   .34   .35   .36
      Z1=C(2)+C(3)                                                   .37
      Z2=C(4)-C(1)                                                   .38
      WRITE(6,14)Z1,Z2                                               .39   .40   .41
      IF (M2-4) 44,70,44                                             .42
   44 WRITE (6,13)                                                   .43   .44
      IF(E(3))50,45,50                                               .45
   45 WRITE(6,8)E(2),E(4)                                            .46   .47   .48
      GO TO 60                                                       .49
   50 WRITE(6,7)E(2),E(3)                                            .50   .51   .52
   60 WRITE(6,9)E(1)                                                 .53   .54   .55
   70 RETURN                                                         .56
      END                                                            .57

BEGIN ASSEMBLY     12.875
```

```
      SUBROUTINE GUES
      DOUBLE PRECISION A, B, C, C1, C2, C3, C4, C5, C6, C7, C8, C9, C10,
     1D, DT, DTMAX, DSMAX, E, E1, E2, E3, E4, E5, E6, E7, E8, E9, E10,
     2F, G, GI, GII, GMU, GMU1, GMUC, H, P, Q, R, S, T, U, V, W, X1, X2,
     3X3, X4, X5, X6, X7, X8, X9, X10, Y1, Y2, Y3, Y4, Y5, Y6, Y7, Y8,
     4Y9, Y10, Z, Z1, Z2, Z3, Z4, Z5, Z6, Z7, Z8, Z9, Z10
      DIMENSION A(50,50), B(50), C(50), E(50), F(50), GI(50), GII(50),
     1X(2000), Y(2000), Z(100), S(50)
      COMMON/SPR/X, Y
      COMMON/DPR/A, B, C, C1, C2, C3, C4, C5, C6, C7, C8, C9, C10, D,
     1DT, DTMAX, DSMAX, E, E1, E2, E3, E4, E5, E6, E7, E8, E9, E10, F,
     2G, GI, GII, GMU, GMU1, GMUC, H, P, Q, R, S, T, U, V, W, X1, X2,
     3X3, X4, X5, X6, X7, X8, X9, X10, Y1, Y2, Y3, Y4, Y5, Y6, Y7, Y8,
     4Y9, Y10, Z, Z1, Z2, Z3, Z4, Z5, Z6, Z7, Z8, Z9, Z10
      COMMON/INTS/I, II, J, K, K1, K2, K3, K4, K5, K6, K7, K8, K9, K10,
     1L, L1, L2, L3, L4, L5, L6, L7, L8, L9, L10, M, M1, M2, M3, M4, M5,
     2M6, M7, M8, M9, M10, N, N1, N2, N3, N4, N5, N6, N7, N8, N9, N10
    1 FORMAT(36H0SUGGESTED CHANGES IN INITIAL VALUES 4D20.13)
    2 FORMAT(53H0EQUATIONS TO BE SOLVED FOR CHANGES IN INITIAL VALUES )
    3 FORMAT(1H0D19.12,4D20.12)
    7 DO 8 I=1, 4                                           .1
      DO 8 J=2, 5                                           .2
      K = I+4*J-4                                           .3
    8 A(I,J) = B(K)                                         .4    .5    .6
      DO 9 I=1, 4                                           .7
    9 A(I,I+1) = A(I,I+1)-1.                                .8    .9
      DO 12 I=1, 4                                          .10
   12 A(I,6) = C(I)-B(I)                                    .11   .12
      IF (L9) 18, 18, 14                                    .13
   14 WRITE (6,2)                                           .14   .15
      WRITE (6,3)((A(I,J),J=2,6),I=1,4)                     .16   .17   .18   .19   .20   .21
                                                           .22   .23
   18 CALL MATR                                             .24
      IF (L9) 25, 25, 20                                    .25
   20 WRITE (6,3)((A(I,J),J=2,6),I=1,9)                     .26   .27   .28   .29   .30   .31
                                                           .32   .33
   25 WRITE (6,1)(Z(I),I=1,4)                               .34   .35   .36   .37   .38
      Z1=0.                                                 .39
      DO 30 I=1,4                                           .40
      Z2=DABS(Z(I))                                         .41
      IF (Z1-Z2) 27,30,30                                   .42
   27 Z1=Z2                                                 .43
   30 CONTINUE                                              .44   .45
      IF (Z1-P) 50,50,35                                    .46
   35 P=.5*P                                                .47
      Z1=P/Z1                                               .48
      DO 40 I=1,4                                           .49
   40 Z(I)=Z(I)*Z1                                          .50   .51
      WRITE (6,1) (Z(I),I=1,4)                              .52   .53   .54   .55   .56
      GO TO 60                                              .57
   50 P=Z1                                                  .58
   60 RETURN                                                .59
      END                                                   .60
```

```
      SUBROUTINE MATR
      DOUBLE PRECISION A, B, C, C1, C2, C3, C4, C5, C6, C7, C8, C9, C10,
     1D, DT, DTMAX, DSMAX, E, E1, E2, E3, E4, E5, E6, E7, E8, E9, E10,
     2F, G, GI, GII, GMU, GMU1, GMUC, H, P, Q, R, S, T, U, V, W, X1, X2,
     3X3, X4, X5, X6, X7, X8, X9, X10, Y1, Y2, Y3, Y4, Y5, Y6, Y7, Y8,
     4Y9, Y10, Z, Z1, Z2, Z3, Z4, Z5, Z6, Z7, Z8, Z9, Z10
      DIMENSION A(50,50), B(50), C(50), E(50), F(50), GI(50), GII(50),
     1X(2000), Y(2000), Z(100), S(50)
      COMMON/SPR/X, Y
      COMMON/DPR/A, B, C, C1, C2, C3, C4, C5, C6, C7, C8, C9, C10, D,
     1DT, DTMAX, DSMAX, E, E1, E2, E3, E4, E5, E6, E7, E8, E9, E10, F,
     2G, GI, GII, GMU, GMU1, GMUC, H, P, Q, R, S, T, U, V, W, X1, X2,
     3X3, X4, X5, X6, X7, X8, X9, X10, Y1, Y2, Y3, Y4, Y5, Y6, Y7, Y8,
     4Y9, Y10, Z, Z1, Z2, Z3, Z4, Z5, Z6, Z7, Z8, Z9, Z10
      COMMON/INTS/I, II, J, K, K1, K2, K3, K4, K5, K6, K7, K8, K9, K10,
     1L, L1, L2, L3, L4, L5, L6, L7, L8, L9, L10, M, M1, M2, M3, M4, M5,
     2M6, M7, M8, M9, M10, N, N1, N2, N3, N4, N5, N6, N7, N8, N9, N10
      K5=5                                                        .1
    6 K5 = K5-1                                                   .2
      IF (K5-1) 35, 35, 7                                         .3
    7 Z1 = 0.                                                     .4
      K6=5-K5                                                     .5
      K2=K6+1                                                     .6
      K4 = K2+1                                                   .7
      DO 15 I=K6,4                                                .8
      Z2 = DABS(A(I,K2))                                          .9
      IF (Z2-Z1) 15, 15, 13                                       .10
   13 Z1 = Z2                                                     .11
      K3 = I                                                      .12
   15 CONTINUE                                                    .13    .14
      DO 17 J=K4, 6                                               .15
   17 A(K2+5,J) = A(K3,J)/A(K3,K2)                                .16    .17
      DO 20 I=K6,4                                                .18
      IF (I-K3) 18, 20, 18                                        .19
   18 DO 19 J=K4, 6                                               .20
      A(I,J) = A(I,J)-A(I,K2)*A(K2+5,J)                           .21
   19 CONTINUE                                                    .22    .23
   20 CONTINUE                                                    .24    .25
      IF (K6-K3) 24, 28, 24                                       .26
   24 DO 25 J=K4, 6                                               .27
   25 A(K3,J) = A(K6,J)                                           .28    .29
   28 GO TO 6                                                     .30
   35 Z(4) = A(4,6)/A(4,5)                                        .31
      DO 40 I=1, 3                                                .32
      K = 10-I                                                    .33
      K1 = 6-I                                                    .34
      Z(K-6) = A(K,6)                                             .35
      DO 40 J=K1, 5                                               .36
   40 Z(K-6) = Z(K-6)-A(K,J)*Z(J-1)                               .37    .38    .39
      RETURN                                                      .40
      END                                                         .41

BEGIN ASSEMBLY    01.056
```

```
      SUBROUTINE ORBIT
      DOUBLE PRECISION A, B, C, C1, C2, C3, C4, C5, C6, C7, C8, C9, C10,
     1D, DT, DTMAX, DSMAX, E, E1, E2, E3, E4, E5, E6, E7, E8, E9, E10,
     2F, G, GI, GII, GMU, GMU1, GMUC, H, P, Q, R, S, T, U, V, W, X1, X2,
     3X3, X4, X5, X6, X7, X8, X9, X10, Y1, Y2, Y3, Y4, Y5, Y6, Y7, Y8,
     4Y9, Y10, Z, Z1, Z2, Z3, Z4, Z5, Z6, Z7, Z8, Z9, Z10
      DIMENSION A(50,50), B(50), C(50), E(50), F(50), GI(50), GII(50),
     1X(2000), Y(2000), Z(100), S(50)
      COMMON/SPR/X, Y
      COMMON/DPR/A, B, C, C1, C2, C3, C4, C5, C6, C7, C8, C9, C10, D,
     1DT, DTMAX, DSMAX, E, E1, E2, E3, E4, E5, E6, E7, E8, E9, E10, F,
     2G, GI, GII, GMU, GMU1, GMUC, H, P, Q, R, S, T, U, V, W, X1, X2,
     3X3, X4, X5, X6, X7, X8, X9, X10, Y1, Y2, Y3, Y4, Y5, Y6, Y7, Y8,
     4Y9, Y10, Z, Z1, Z2, Z3, Z4, Z5, Z6, Z7, Z8, Z9, Z10
      COMMON/INTS/I, II, J, K, K1, K2, K3, K4, K5, K6, K7, K8, K9, K10,
     1L, L1, L2, L3, L4, L5, L6, L7, L8, L9, L10, M, M1, M2, M3, M4, M5,
     2M6, M7, M8, M9, M10, N, N1, N2, N3, N4, N5, N6, N7, N8, N9, N10
    1 FORMAT (1H02D21.14,D28.14,D21.14,D28.14)
    5 CALL TOWER                                                   .1
      CALL SETDT                                                   .2
      T=T+DT                                                       .3
  511 CALL SETPR                                                   .4
      IF (N4) 7, 7, 6                                              .5
    6 CALL PRNT                                                    .6
    7 IF (N9) 11, 11, 8                                            .7
    8 IF (N5) 5, 5, 12                                             .8
   11 M2=4                                                         .9
      L10=0                                                        .10
   12 IF (M2-4) 13,14,13                                           .11
   13 CALL ENDOR                                                   .12
   14 IF (L10) 20,20,15                                            .13
   15 Z1=0.                                                        .14
      DO 152 I=1,4                                                 .15
      Z2= DABS(C(I)-B(I))                                          .16
      IF (Z1-Z2) 150,152,152                                       .17
  150 Z1=Z2                                                        .18
  152 CONTINUE                                                     .19    .20
      IF (Z1-C7) 20,20,155                                         .21
  155 M1=0                                                         .22
      DO 157 I=1,4                                                 .23
  157 C(I)=Z(I)    +C(I)                                           .24    .25
   16 L10=L10-1                                                    .26
      RETURN                                                       .27
   20 M1 = 1                                                       .28
      GO TO 16                                                     .29
      END                                                          .30

BEGIN ASSEMBLY    14.886
```

```
      SUBROUTINE POWER
      DOUBLE PRECISION A, B, C, C1, C2, C3, C4, C5, C6, C7, C8, C9, C10,
     1D, DT, DTMAX, DSMAX, E, E1, E2, E3, E4, E5, E6, E7, E8, E9, E10,
     2F, G, GI, GII, GMU, GMU1, GMUC, H, P, Q, R, S, T, U, V, W, X1, X2,
     3X3, X4, X5, X6, X7, X8, X9, X10, Y1, Y2, Y3, Y4, Y5, Y6, Y7, Y8,
     4Y9, Y10, Z, Z1, Z2, Z3, Z4, Z5, Z6, Z7, Z8, Z9, Z10
      DIMENSION A(50,50), B(50), C(50), E(50), F(50), GI(50), GII(50),
     1X(2000), Y(2000), Z(100), S(50)
      COMMON/SPR/X, Y
      COMMON/DPR/A, B, C, C1, C2, C3, C4, C5, C6, C7, C8, C9, C10, D,
     1DT, DTMAX, DSMAX, E, E1, E2, E3, E4, E5, E6, E7, E8, E9, E10, F,
     2G, GI, GII, GMU, GMU1, GMUC, H, P, Q, R, S, T, U, V, W, X1, X2,
     3X3, X4, X5, X6, X7, X8, X9, X10, Y1, Y2, Y3, Y4, Y5, Y6, Y7, Y8,
     4Y9, Y10, Z, Z1, Z2, Z3, Z4, Z5, Z6, Z7, Z8, Z9, Z10
      COMMON/INTS/I, II, J, K, K1, K2, K3, K4, K5, K6, K7, K8, K9, K10,
     1L, L1, L2, L3, L4, L5, L6, L7, L8, L9, L10, M, M1, M2, M3, M4, M5,
     2M6, M7, M8, M9, M10, N, N1, N2, N3, N4, N5, N6, N7, N8, N9, N10
      DO 10 I=1, M2                                                  .1
   10 A(I,1) = B(I)                                                  .2     .3
      Z1 = B(1)+GMU                                                  .4
      A(21,1) = Z1*Z1                                                .5
      A(22,1) = A(21,1)+B(2)*B(2)                                    .6
      A(23,1) = 1./A(22,1)                                           .7
      A(24,1) = 1./(A(22,1)+1.-Z1-Z1)                                .8
      A(25,1) = -GMU1*A(23,1)*DSQRT(A(23,1))                         .9
      A(26,1) = GMU*A(24,1)*DSQRT(A(24,1))                           .10
      A(50,1) = A(25,1)+A(26,1)                                      .11
      GO TO (20,14), N2                                              .12
   14 A(27,1) = Z1*B(2)                                              .13
      A(28,1) = 1.-3.*A(23,1)*A(21,1)                                .14
      A(29,1) = 1.-3.*A(24,1)*(A(21,1)+1.-Z1-Z1)                     .15
      A(30,1) = A(23,1)*A(27,1)                                      .16
      A(31,1) = A(24,1)*(A(27,1)-B(2))                               .17
      A(32,1) = A(25,1)*A(28,1)+A(26,1)*A(29,1)                      .18
      A(33,1) =(A(25,1)*A(30,1)+A(26,1)*A(31,1))*3.D0                .19
   20 DO 50 N=1, L                                                   .20
      L1 = N+1                                                       .21
      L2 = N+2                                                       .22
      Z1 = 0.                                                        .23
      Z2 = 0.                                                        .24
      Z3 = 0.                                                        .25
      A(1,N+1) = GII(N)*(A(3,N)+A(2,N))                              .26
      A(2,N+1) = GII(N)*(A(4,N)-A(1,N))                              .27
      DO 24 J=1, L1                                                  .28
      K = L2-J                                                       .29
      Z1 = Z1+A(1,J)*A(1,K)                                          .30
      Z2 = Z2+A(1,J)*A(2,K)                                          .31
   24 Z3 = Z3+A(2,J)*A(2,K)                                          .32    .33
      A(21,L1) = 2.*GMU*A(1,L1)+Z1                                   .34
      A(22,L1) = A(21,L1)+Z3                                         .35
      Z4 = 0.                                                        .36
      Z5 = 0.                                                        .37
      Z6 = 0.                                                        .38
```

```
      SUBROUTINE POWER
      DOUBLE PRECISION A, B, C, C1, C2, C3, C4, C5, C6, C7, C8, C9, C10,
     1D, DT, DTMAX, DSMAX, E, E1, E2, E3, E4, E5, E6, E7, E8, E9, E10,
     2F, G, GI, GII, GMU, GMU1, GMUC, H, P, Q, R, S, T, U, V, W, X1, X2,
     3X3, X4, X5, X6, X7, X8, X9, X10, Y1, Y2, Y3, Y4, Y5, Y6, Y7, Y8,
     4Y9, Y10, Z, Z1, Z2, Z3, Z4, Z5, Z6, Z7, Z8, Z9, Z10
      DIMENSION A(50,50), B(50), C(50), E(50), F(50), GI(50), GII(50),
     1X(2000), Y(2000), Z(100), S(50)
      COMMON/SPR/X, Y
      COMMON/DPR/A, B, C, C1, C2, C3, C4, C5, C6, C7, C8, C9, C10, D,
     1DT, DTMAX, DSMAX, E, E1, E2, E3, E4, E5, E6, E7, E8, E9, E10, F,
     2G, GI, GII, GMU, GMU1, GMUC, H, P, Q, R, S, T, U, V, W, X1, X2,
     3X3, X4, X5, X6, X7, X8, X9, X10, Y1, Y2, Y3, Y4, Y5, Y6, Y7, Y8,
     4Y9, Y10, Z, Z1, Z2, Z3, Z4, Z5, Z6, Z7, Z8, Z9, Z10
      COMMON/INTS/I, II, J, K, K1, K2, K3, K4, K5, K6, K7, K8, K9, K10,
     1L, L1, L2, L3, L4, L5, L6, L7, L8, L9, L10, M, M1, M2, M3, M4, M5,
     2M6, M7, M8, M9, M10, N, N1, N2, N3, N4, N5, N6, N7, N8, N9, N10
      DO 10 I=1, M2                                                      .1
   10 A(I,1) = B(I)                                                      .2     .3
      Z1 = B(1)+GMU                                                      .4
      A(21,1) = Z1*Z1                                                    .5
      A(22,1) = A(21,1)+B(2)*B(2)                                        .6
      A(23,1) = 1./A(22,1)                                               .7
      A(24,1) = 1./(A(22,1)+1.-Z1-Z1)                                    .8
      A(25,1) = -GMU1*A(23,1)*DSQRT(A(23,1))                             .9
      A(26,1) = GMU*A(24,1)*DSQRT(A(24,1))                               .10
      A(50,1) = A(25,1)+A(26,1)                                          .11
      GO TO (20,14), N2                                                  .12
   14 A(27,1) = Z1*B(2)                                                  .13
      A(28,1) = 1.-3.*A(23,1)*A(21,1)                                    .14
      A(29,1) = 1.-3.*A(24,1)*(A(21,1)+1.-Z1-Z1)                         .15
      A(30,1) = A(23,1)*A(27,1)                                          .16
      A(31,1) = A(24,1)*(A(27,1)-B(2))                                   .17
      A(32,1) = A(25,1)*A(28,1)+A(26,1)*A(29,1)                          .18
      A(33,1) =(A(25,1)*A(30,1)+A(26,1)*A(31,1))*3.D0                    .19
   20 DO 50 N=1, L                                                       .20
      L1 = N+1                                                           .21
      L2 = N+2                                                           .22
      Z1 = 0.                                                            .23
      Z2 = 0.                                                            .24
      Z3 = 0.                                                            .25
      A(1,N+1) = GII(N)*(A(3,N)+A(2,N))                                  .26
      A(2,N+1) = GII(N)*(A(4,N)-A(1,N))                                  .27
      DO 24 J=1, L1                                                      .28
      K = L2-J                                                           .29
      Z1 = Z1+A(1,J)*A(1,K)                                              .30
      Z2 = Z2+A(1,J)*A(2,K)                                              .31
   24 Z3 = Z3+A(2,J)*A(2,K)                                              .32    .33
      A(21,L1) = 2.*GMU*A(1,L1)+Z1                                       .34
      A(22,L1) = A(21,L1)+Z3                                             .35
      Z4 = 0.                                                            .36
      Z5 = 0.                                                            .37
      Z6 = 0.                                                            .38
```

```
      SUBROUTINE PRNT
      DOUBLE PRECISION A, B, C, C1, C2, C3, C4, C5, C6, C7, C8, C9, C10,
     1D, DT, DTMAX, DSMAX, E, E1, E2, E3, E4, E5, E6, E7, E8, E9, E10,
     2F, G, GI, GII, GMU, GMU1, GMUC, H, P, Q, R, S, T, U, V, W, X1, X2,
     3X3, X4, X5, X6, X7, X8, X9, X10, Y1, Y2, Y3, Y4, Y5, Y6, Y7, Y8,
     4Y9, Y10, Z, Z1, Z2, Z3, Z4, Z5, Z6, Z7, Z8, Z9, Z10
      DIMENSION A(50,50), B(50), C(50), E(50), F(50), GI(50), GII(50),
     1X(2000), Y(2000), Z(100), S(50)
      COMMON/SPR/X, Y
      COMMON/DPR/A, B, C, C1, C2, C3, C4, C5, C6, C7, C8, C9, C10, D,
     1DT, DTMAX, DSMAX, E, E1, E2, E3, E4, E5, E6, E7, E8, E9, E10, F,
     2G, GI, GII, GMU, GMU1, GMUC, H, P, Q, R, S, T, U, V, W, X1, X2,
     3X3, X4, X5, X6, X7, X8, X9, X10, Y1, Y2, Y3, Y4, Y5, Y6, Y7, Y8,
     4Y9, Y10, Z, Z1, Z2, Z3, Z4, Z5, Z6, Z7, Z8, Z9, Z10
      COMMON/INTS/I, II, J, K, K1, K2, K3, K4, K5, K6, K7, K8, K9, K10,
     1L, L1, L2, L3, L4, L5, L6, L7, L8, L9, L10, M, M1, M2, M3, M4, M5,
     2M6, M7, M8, M9, M10, N, N1, N2, N3, N4, N5, N6, N7, N8, N9, N10
    1 FORMAT(1H0D15.8,4D20.12,D23.15)
    2 FORMAT(5H     4D22.14,D23.15)
      N4 = 0                                                          .1
      Z2 = B(2)+B(3)                                                  .2
      Z3 = B(4)-B(1)                                                  .3
      Z1 = B(2)*B(2)                                                  .4
      Z4 = (B(1)+GMU)**2+Z1                                           .5
      Z5 = (B(1) GMU1)**2+Z1                                          .6
      Z6 = DSQRT(Z4)                                                  .7
      Z7 = DSQRT(Z5)                                                  .8
      Z9 = GMU*(2./Z7+Z5)-GMU1*(2./Z6+Z4)-Z2*Z2-Z3*Z3                 .9
      WRITE (6,1)T, B(1), B(2), Z2, Z3, Z9                            .10   .11   .12
      L3 = L3+1                                                       .13
      X(L3) = B(1)                                                    .14
      Y(L3) = B(2)                                                    .15
      GO TO (20,6), N7                                                .16
    6 Z4 = 1./(Z4*Z6)                                                 .17
      Z5 = 1./(Z5*Z7)                                                 .18
      Z1 = GMU*Z5-GMU1*Z4                                             .19
      Z6 = B(4)-Z1*B(1)+GMU*GMU1*(Z4-Z5)                              .20
      Z7 = -B(3)-Z1*B(2)                                              .21
      DO 8 I=1, 13, 4                                                 .22
      Z8 = -Z6*B(I+4)-Z7*B(I+5)+Z2*B(I+6)+Z3*B(I+7)                   .23
    8 WRITE (6,2)B(I+4), B(I+5), B(I+6), B(I+7), Z8                   .24   .25   .26   .27
   20 RETURN                                                          .28
      END                                                             .29

BEGIN ASSEMBLY    05.619
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```
      SUBROUTINE RE36D
      DOUBLE PRECISION A, B, C, C1, C2, C3, C4, C5, C6, C7, C8, C9, C10,
     1D, DT, DTMAX, DSMAX, E, E1, E2, E3, E4, E5, E6, E7, E8, E9, E10,
     2F, G, GI, GII, GMU, GMU1, GMUC, H, P, Q, R, S, T, U, V, W, X1, X2,
     3X3, X4, X5, X6, X7, X8, X9, X10, Y1, Y2, Y3, Y4, Y5, Y6, Y7, Y8,
     4Y9, Y10, Z, Z1, Z2, Z3, Z4, Z5, Z6, Z7, Z8, Z9, Z10
      DIMENSION A(50,50), B(50), C(50), E(50), F(50), GI(50), GII(50),
     1X(2000), Y(2000), Z(100), S(50)
      COMMON/SPR/X, Y
      COMMON/DPR/A, B, C, C1, C2, C3, C4, C5, C6, C7, C8, C9, C10, D,
     1DT, DTMAX, DSMAX, E, E1, E2, E3, E4, E5, E6, E7, E8, E9, E10, F,
     2G, GI, GII, GMU, GMU1, GMUC, H, P, Q, R, S, T, U, V, W, X1, X2,
     3X3, X4, X5, X6, X7, X8, X9, X10, Y1, Y2, Y3, Y4, Y5, Y6, Y7, Y8,
     4Y9, Y10, Z, Z1, Z2, Z3, Z4, Z5, Z6, Z7, Z8, Z9, Z10
      COMMON/INTS/I, II, J, K, K1, K2, K3, K4, K5, K6, K7, K8, K9, K10,
     1L, L1, L2, L3, L4, L5, L6, L7, L8, L9, L10, M, M1, M2, M3, M4, M5,
     2M6, M7, M8, M9, M10, N, N1, N2, N3, N4, N5, N6, N7, N8, N9, N10
    1 FORMAT(14I5)
    2 FORMAT(7D10.2)
    3 FORMAT(7D14.6)
    4 FORMAT (2D23.16)
    5 FORMAT (3D24.16)
      READ  (5,5) GMU,C(1),C(2),C5,C(3),C(4)                              ,1    ,2    ,3
      READ (5,1)M,M2,M3,M4, M5, M6, M7, M8, M9, M10, L10, L9, L8, L7      ,4    ,5    ,6    ,7    ,8    ,9
                                                                          ,10   ,11   ,12   ,13   ,14   ,15
                                                                          ,16   ,17   ,18   ,19
      READ (5,2)C1, C2, C3, C4, C6, C7, C8, C9, C10, DTMAX, DSMAX         ,20   ,21   ,22
      WRITE (6,5) GMU,C(1),C(2),C5,C(3),C(4)                              ,23   ,24   ,25
      WRITE (6,1)M,M2,M3,M4, M5, M6, M7, M8, M9, M10, L10, L9, L8, L7     ,26   ,27   ,28
      WRITE (6,3)C1, C2, C3, C4, C6, C7, C8, C9, C10, DTMAX, DSMAX        ,29   ,30   ,31
      RETURN                                                              ,32
      END                                                                 ,33

 BEGIN ASSEMBLY    12.879
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```
      SUBROUTINE SETDT
      DOUBLE PRECISION A, B, C, C1, C2, C3, C4, C5, C6, C7, C8, C9, C10,
     1D, DT, DTMAX, DSMAX, E, E1, E2, E3, E4, E5, E6, E7, E8, E9, E10,
     2F, G, GI, GII, GMU, GMU1, GMUC, H, P, Q, R, S, T, U, V, W, X1, X2,
     3X3, X4, X5, X6, X7, X8, X9, X10, Y1, Y2, Y3, Y4, Y5, Y6, Y7, Y8,
     4Y9, Y10, Z, Z1, Z2, Z3, Z4, Z5, Z6, Z7, Z8, Z9, Z10
      DIMENSION A(50,50), B(50), C(50), E(50), F(50), GI(50), GII(50),
     1X(2000), Y(2000), Z(100), S(50)
      COMMON/SPR/X, Y
      COMMON/DPR/A, B, C, C1, C2, C3, C4, C5, C6, C7, C8, C9, C10, D,
     1DT, DTMAX, DSMAX, E, E1, E2, E3, E4, E5, E6, E7, E8, E9, E10, F,
     2G, GI, GII, GMU, GMU1, GMUC, H, P, Q, R, S, T, U, V, W, X1, X2,
     3X3, X4, X5, X6, X7, X8, X9, X10, Y1, Y2, Y3, Y4, Y5, Y6, Y7, Y8,
     4Y9, Y10, Z, Z1, Z2, Z3, Z4, Z5, Z6, Z7, Z8, Z9, Z10
      COMMON/INTS/I, II, J, K, K1, K2, K3, K4, K5, K6, K7, K8, K9, K10,
     1L, L1, L2, L3, L4, L5, L6, L7, L8, L9, L10, M, M1, M2, M3, M4, M5,
     2M6, M7, M8, M9, M10, N, N1, N2, N3, N4, N5, N6, N7, N8, N9, N10
      IF (T+D-C5) 2, 6, 6                                                 ,1
    2 DT = D                                                              ,2
    3 CALL TAYL                                                           ,3
      RETURN                                                              ,4
    6 DT = C5-T                                                           ,5
      N5 = 1                                                              ,6
      GO TO 3                                                             ,7
      END                                                                 ,8

BEGIN ASSEMBLY    03.862
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      SUBROUTINE SETPR
      DOUBLE PRECISION A, B, C, C1, C2, C3, C4, C5, C6, C7, C8, C9, C10,
     1D, DT, DTMAX, DSMAX, E, E1, E2, E3, E4, E5, E6, E7, E8, E9, E10,
     2F, G, GI, GII, GMU, GMU1, GMUC, H, P, Q, R, S, T, U, V, W, X1, X2,
     3X3, X4, X5, X6, X7, X8, X9, X10, Y1, Y2, Y3, Y4, Y5, Y6, Y7, Y8,
     4Y9, Y10, Z, Z1, Z2, Z3, Z4, Z5, Z6, Z7, Z8, Z9, Z10
      DIMENSION A(50,50), B(50), C(50), E(50), F(50), GI(50), GII(50),
     1X(2000), Y(2000)  Z(100), S(50)
      COMMON/SPR/X, Y
      COMMON/DPR/A, B, C, C1, C2, C3, C4, C5, C6, C7, C8, C9, C10, D,
     1DT, DTMAX, DSMAX, E, E1, E2, E3, E4, E5, E6, E7, E8, E9, E10, F,
     2G, GI, GII, GMU, GMU1, GMUC, H, P, Q, R, S, T, U, V, W, X1, X2,
     3X3, X4, X5, X6, X7, X8, X9, X10, Y1, Y2, Y3, Y4, Y5, Y6, Y7, Y8,
     4Y9, Y10, Z, Z1, Z2, Z3, Z4, Z5, Z6, Z7, Z8, Z9, Z10
      COMMON/INTS/I, II, J, K, K1, K2, K3, K4, K5, K6, K7, K8, K9, K10,
     1L, L1, L2, L3, L4, L5, L6, L7, L8, L9, L10, M, M1, M2, M3, M4, M5,
     2M6, M7, M8, M9, M10, N, N1, N2, N3, N4, N5, N6, N7, N8, N9, N10
      N3 = N3+1                                                            .1
      IF (N3-M3) 2, 5, 5                                                   .2
    2 N4 = 0                                                               .3
    3 RETURN                                                               .4
    5 N3 = 0                                                               .5
      N4 = 1                                                               .6
      GO TO 3                                                              .7
      END                                                                  .8

BEGIN ASSEMBLY    03.864
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      SUBROUTINE SETUP
      DOUBLE PRECISION A, B, C, C1, C2, C3, C4, C5, C6, C7, C8, C9, C10,
     1D, DT, DTMAX. DSMAX. E. E1. E2. E3, E4, E5, E6, E7, E8, E9, E10,
     2F, G, GI, GII, GMU, GMU1, GMUC, H, P, Q, R, S, T, U, V, W, X1, X2,
     3X3, X4, X5, X6, X7, X8, X9, X10, Y1, Y2, Y3, Y4, Y5, Y6, Y7, Y8,
     4Y9, Y10, Z, Z1, Z2, Z3, Z4, Z5, Z6, Z7, Z8, Z9, Z10
      DIMENSION A(50,50), B(50), C(50), E(50), F(50), GI(50), GII(50),
     1X(2000), Y(2000), Z(100), S(50)
      COMMON/SPR/X, Y
      COMMON/DPR/A, B, C, C1, C2, C3, C4, C5, C6, C7, C8, C9, C10, D,
     1DT, DTMAX, DSMAX, E, E1, E2, E3, E4, E5, E6, E7, E8, E9, E10, F,
     2G, GI, GII, GMU, GMU1, GMUC, H, P, Q, R, S, T, U, V, W, X1, X2,
     3X3, X4, X5, X6, X7, X8, X9, X10, Y1, Y2, Y3, Y4, Y5, Y6, Y7, Y8,
     4Y9, Y10, Z, Z1, Z2, Z3, Z4, Z5, Z6, Z7, Z8, Z9, Z10
      COMMON/INTS/I. II, J, K, K1. K2, K3, K4, K5, K6, K7, K8, K9, K10,
     1L, L1, L2, L3, L4, L5, L6, L7, L8, L9, L10, M, M1, M2, M3, M4, M5,
     2M6, M7, M8, M9, M10, N, N1, N2, N3, N4, N5, N6, N7, N8, N9, N10
    2 FORMAT(112H1         T                    X                    Y
     1          DX/DT                    DY/DT                    C  )
      T = 0.                                                               .1
      N7 = M7                                                              .2
      L = M-1                                                              .3
      L3 = 0                                                               .4
      N3 = 0                                                               .5
      N5 = 0                                                               .6
      N9 = M9                                                              .7
      S(1) = 1.                                                            .8
      GMU1 = GMU-1.                                                        .9
      GMUC = 1.-GMU-GMU                                                    .10
      DO 7 I=1, 4                                                          .11
    7 B(I) = C(I)                                                          .12    .13
      DO 8 I=1, M                                                          .14
      GI(I) = 1                                                            .15
    8 GII(I) = 1./GI(I)                                                    .16    .17
      N2 = 1                                                               .18
      IF (M2-4) 20, 20, 9                                                  .19
    9 N2 = 2                                                               .20
   14 DO 15 I=6, 19                                                        .21
   15 B(I) = 0.                                                            .22    .23
      DO 16 I=5, 20, 5                                                     .24
   16 B(I) = 1.                                                            .25    .26
   20 WRITE (6,2)                                                          .27    .28
      CALL PRNT                                                            .29
      C(5)=Z9                                                              .30
      RETURN                                                               .31
      END                                                                  .32

BEGIN ASSEMBLY    05.623
```

```
      SUBROUTINE TAYL
      DOUBLE PRECISION A, B, C, C1, C2, C3, C4, C5, C6, C7, C8, C9, C10,
     1D, DT, DTMAX, DSMAX, E, E1, E2, E3, E4, E5, E6, E7, E8, E9, E10,
     2F, G, GI, GII, GMU, GMU1, GMUC, H, P, Q, R, S, T, U, V, W, X1, X2,
     3X3, X4, X5, X6, X7, X8, X9, X10, Y1, Y2, Y3, Y4, Y5, Y6, Y7, Y8,
     4Y9, Y10, Z, Z1, Z2, Z3, Z4, Z5, Z6, Z7, Z8, Z9, Z10
      DIMENSION A(50,50), B(50), C(50), E(50), F(50), GI(50), GII(50),
     1X(2000), Y(2000), Z(100), S(50)
      COMMON/SPR/X, Y
      COMMON/DPR/A, B, C, C1, C2, C3, C4, C5, C6, C7, C8, C9, C10, D,
     1DT, DTMAX, DSMAX, E, E1, E2, E3, E4, E5, E6, E7, E8, E9, E10, F,
     2G, GI, GII, GMU, GMU1, GMUC, H, P, Q, R, S, T, U, V, W, X1, X2,
     3X3, X4, X5, X6, X7, X8, X9, X10, Y1, Y2, Y3, Y4, Y5, Y6, Y7, Y8,
     4Y9, Y10, Z, Z1, Z2, Z3, Z4, Z5, Z6, Z7, Z8, Z9, Z10
      COMMON/INTS/I, II, J, K, K1, K2, K3, K4, K5, K6, K7, K8, K9, K10,
     1L, L1, L2, L3, L4, L5, L6, L7, L8, L9, L10, M, M1, M2, M3, M4, M5,
     2M6, M7, M8, M9, M10, N, N1, N2, N3, N4, N5, N6, N7, N8, N9, N10
      DO 2 I=2, M                                                        .1
    2 S(I) = S(I-1)*DT                                                   .2      .3
      DO 8 I=1, M2                                                       .4
      B(I) = A(I,1)                                                      .5
      DO 8 J=2, M                                                        .6
    8 B(I) = B(I)+A(I,J)*S(J)                                            .7      .8      .9
      RETURN                                                             .10
      END                                                                .11

BEGIN ASSEMBLY    03.869
```

SAMPLE PROBLEM COMPUTER OUTPUT

Rabe's parameter $d_0 = 1.01$

(First 5 lines are the print-out of the input variables)

```
0.953875357107086 8D-03  0.5040461246428929D 00 -0.874685658000[illegible] 00
0.7859711486899776D 02  0.8617042917355127D 00  0.496561161059981[illegible]-00
 16   20   20    1    1    1    2    0  200    0    3    0    0    0
0.1000000D-00  0.1000000D-00  0.2000000D-00  0.3000000D-00  0.2990000D 01  1.0000000D-11 -0.
-0.            -0.            0.1300000D 02  0.1000000D 01
```

```
       T                    X                    Y                    DX/DT                  DY/DT                    C

0.                 0.5040461246643D 00 -0.8746856580000D 00 -0.1298136626645D-01 -0.7484963582910-02  0.3000073267357210 01
     1.000000000000000D 00  0.                   0.                   0.                   -0.7345780913094440-02
     0.                     1.000000000000000 00 0.                   0.                    0.1273095463054660-01
     0.                     0.                   1.000000000000000D 00 0.                  -0.1298136626448730-01
     0.                     0.                   0.                   1.000000000000000D 00 -0.7484963582911010-02

0.15454830D 02  C 3361477013940-00 -0.9458208677430 00 -0.6105015616570-02 -0.1508066037750-02  0.3000073267357210 01
     -0.1857197078'969D 02 -0.8006613194839 30 01  0.5530534425966950 01 -0.1869365824266 8D 02 -0.7345780913094040-02
      0.2830151716194 8D 02  0.1161454954737 6D 02 -0.9515901007972 9D 01  0.2964111474206 6D 02  0.1273095463054610-01
     -0.3002810792934 4D 02 -0.1351292648793 2D 02  0.8955711164543 9D 01 -0.3170625892430 8D 02 -0.1298136626448680-01
     -0.1454888200873 5D 02 -0.6343014782044 6D 01  0.4936111432156 0D 01 -0.1620964718742 7D 02 -0.7484963582910680-02

0.30847312D 02  0.35244565903 7D-00 -0.9291844123230 00  0.7972571555400-02  0.3632901049620-02  0.3000073267357210 01
     -0.1858977843757 4D 02 -0.5603185001632 2D 01  0.8111469584989 9D 01 -0.1801985094295 5D 02 -0.7345780913094880-02
      0.3539568004028 9D 02  0.1252210244314 2D 02 -0.1360963710126 0D 02  0.3418333960242 4D 02  0.1273095463054940-01
     -0.3558478523430 7D 02 -0.1108792940350 9D 02  0.1480500875394 1D 02 -0.3344165228317 3D 02 -0.1298136626449060-01
     -0.2301361285533 6D 02 -0.8331887774360 2D 01  0.8497755895683 4D 01 -0.2167277259854 4D 02 -0.7484963582914010-02

0.46212424D 02  0.5303152797000 00 -0.8358503183590 00  0.1232977501140-01  0.7603546792130-02  0.3000073267357210 01
     -0.1534277545692 8D 01 -0.9249897565410 8D 00  0.5324870658959 8D 00 -0.1727535081642 4D 01 -0.7345780913092910-02
      0.3342890956374 6D 01  0.5262803753096 6D-01 -0.2798057366707 1D 01  0.2815022712444 8D 01  0.1273095463054780-01
     -0.1160619981173 2D 01 -0.2198335465160 7D-00  0.8207359599849 1D 00 -0.2184880184228 7D 01 -0.1298136626448640-01
     -0.2039111297456 0D 01  0.1881489710336 6D-00  0.2197730081301 4D 01 -0.2120124640855 0D 01 -0.7484963582912790-02

0.61676668D 02  0.6398146161410 00 -0.7687834405020 00 -0.2428344688350-03 -0.1429316227510-02  0.3000073267357210 01
      0.1144932120460 6D 02  0.1099415091082 4D 02 -0.9307864386541 5D 01  0.1284651884527 6D 02 -0.7345780913094720-02
     -0.1984348836523 3D 02 -0.1681560802433 4D 02  0.1707727252400 7D 02 -0.2063321733400 1D 02  0.1273095463054740-01
      0.1875706645883 3D 02  0.1746469974843 1D 02 -0.1613554446102 7D 02  0.2142169930292 5D 02 -0.1298136626448930-01
      0.1083407293368 0D 02  0.8865500782466 7D 01 -0.9885823141583 4D 01  0.1148198772220 0D 02 -0.7484963582912680-02

0.77173481D 02  0.5223430493470 00 -0.8638003930790 00 -0.1269306027860-01 -0.7789218513440-02  0.3000073267357210 01
      0.4285502413383 3D 01  0.4241039355710 8D-00 -0.2846729711661 2D 01  0.2124321430754 4D 01 -0.7345780913095200-02
     -0.8003908973956 5D 01 -0.3455348412774 4D 01  0.4232796324595 0D 01 -0.6354489635115 6D 01  0.1273095463054860-01
      0.8935611752315 3D 01  0.2812093789992 6D 01 -0.4848288347838 8D 01  0.5455905634341 2D 01 -0.1298136626449060-01
      0.6488861178791 2D 01  0.3833487223239 3D 01 -0.2974310149467 3D 01  0.5701287511759 0 7D 01 -0.7484963582913660-02

0.78597115D 02  0.5040461925250 00 -0.8746856006660 00 -0.1298133768800-01 -0.7484950691410-02  0.3000073267357210 01
      0.1398050221303 1D-00 -0.1862271031194 7D 01 -0.9750145526547 8D 00 -0.6322908569127 8D 00 -0.7345780913093800-02
     -0.5229055749618 9D 01 -0.1179864336200 9D-00  0.3320479307010 0D 01 -0.2528518907217 4D 01  0.1273095463054670-01
      0.3093884864133 1D 01 -0.2407765519436 6D 01 -0.3129599424302 4D 01  0.3041497478328 1D-01 -0.1298136626448770-01
      J.5736629487001 2D 01  0.1166917927623 1D 01 -0.3340815491374 4D 01  0.3148878915842 6D 01 -0.7484963582912280-02

SIX VARIATIONAL EQUATION CHECKS -0.257550-11-0.472510-12-0.228440-11-0.175730-11 0.119020-11 0.176210-11

CHARACTERISTIC EXPONENTS                                                                                     TRACE

-0.9794509600147 8D 00 0.2016824655891 8D-00        -0.9794509600147 8D 00-0.2016824655891 8D-00        0.4109807997044 6D-01

SUGGESTED CHANGES IN INITIAL VALUES 0.855093940533 2D-08 0.762060534603 3D-07 0.821790465824 3D-08 0.544804442602 8D-07
```

```
       T                        X                        Y                        DX/DT                        DY/DT                        C

0.               0.5040461331940 00 -0.8746855817940 00 -0.1298128184050-01 -0.7484917653410-02  0.3000073266571410 01
     1.000000000000000 00  0.                    C.                     0.                    -0.7345737419030960-02
     0.                    1.000000000000000 00  0.                     0.                     0.1273088115018040-01
     0.                    0.                    1.000000000000000 00   0.                    -0.1298128184052930-01
     0.                    0.                    0.                     1.000000000000000 00 -0.7484917653406200-02

0.154548270 02  0.3361486781860-00 -0.9458205032180 00 -0.6104993558500-02 -0.1508070364960-02  0.3000073266571410 01
    -0.1857195418931150 02 -0.8006629992817770 01  0.5530550821056600 01 -0.1869364673559800 02 -0.7345737419031190-02
     0.2830149729686200 02  0.1161457142603390 02 -0.9515929813147000 01  0.2964109279689600 02  0.1273088115018040-01
    -0.3002807353691900 02 -0.1351294800913300 02  0.8955739313797300 01 -0.3170623223252500 02 -0.1298128184052870-01
    -0.1454886903237700 02 -0.6343024232844100 01  0.4936128643128600 01 -0.1620963499260500 02 -0.7484917653405470-02

UNDRFLOW AT 33302 IN MQ

UNDRFLOW AT 32707 IN MQ

UNDRFLOW AT 32721 IN MQ

UNDRFLOW AT 33312 IN MQ

UNDRFLOW AT 32707 IN MQ

0.308473090 02  0.3524466262100-00 -0.9291840624140 00  0.7972547287340-02  0.3632903103250-02  0.3000073266571410 01
    -0.1858973938067600 02 -0.5603188513566700 01  0.8111475685398300 01 -0.1801980472026100 02 -0.7345737419032340-02
     0.3539560098595800 02  0.1252211224984700 02 -0.1360964594501400 02  0.3418325679692900 02  0.1273088115017930-01
    -0.3558470727294400 02 -0.1108793907764500 02  0.1480501573532500 02 -0.3344156504919000 02 -0.1298128184052960-01
    -0.2301356614844400 02 -0.8331897892309500 01  0.8497761729256000 01 -0.2167272550139600 02 -0.7484917653404290-02

0.462124260 02  0.5303154282100 00 -0.8358502968490 00  0.1232969665410-01  0.7603492008720-02  0.3000073266571410 01
    -0.1534209140267500 01 -0.9249468231040700 00  0.5324423709918000 00 -0.1727464291791300 01 -0.7345737419032790-02
     0.3342764550114700 01  0.5255103048583100-01 -0.2797975340736200 01  0.2814896838278900 01  0.1273088115017950-01
    -0.1160495849445800 01 -0.2197586332372000-00  0.8206536790889200 00 -0.2184755226031400 01 -0.1298128184052960-01
    -0.2039034869151400 01  0.1881938893398900-00  0.2197677910295100 01 -0.2120050334507200 01 -0.7484917553404280-02

0.616766660 02  0.6398139466730 00 -0.7687839988610 00 -0.2428428107110-03 -0.1429305650930-02  0.3000073266571410 01
     0.1144934046082200 02  0.1099414434392700 02 -0.9307862839160100 01  0.1284653527317700 02 -0.7345737419033890-02
    -0.1984352894701800 02 -0.1681560109950600 02  0.1707727608138100 02 -0.2063324874438400 02  0.1273088115017880-01
     0.1875710028572100 02  0.1746468603852400 02 -0.1613554369325000 02  0.2142172363814500 02 -0.1298128184053050-01
     0.1083410011736800 02  0.8865498409943200 01 -0.9885829071331000 01  0.1148200705626800 02 -0.7484917653403330-02

0.771734830 02  0.5223428835410 00 -0.8638004327490 00 -0.1269299208030-01 -0.7789182533460-02  0.3000073266571410 01
     0.4285477141877700 01  0.4240867458963800-00 -0.2846715524960600 01  0.2124297221739300 01 -0.7345737419034120-02
    -0.8003869725941400 01 -0.3455321234779700 01  0.4232772272930900 01 -0.6354449742077200 01  0.1273088115018090-01
     0.8935568868027400 01  0.2812063012264400 01 -0.4848263849667300 01  0.5455862394358800 01 -0.1298128184053190-01
     0.6488840238739700 01  0.3833471052529500 01 -0.2974296827125100 01  0.5701264538675000 01 -0.7484917653405310-02

0.785971150 02  0.5040461331940 00 -0.8746855817940 00 -0.1298128184050-01 -0.7484917653420-02  0.3000073266571410 01
     0.1397824219019000-00 -0.1862284205412800 01 -0.9750019249860100 00 -0.6323131223170300 00 -0.7345737419033180-02
    -0.5229016352731300 01 -0.1179634959038200-00  0.3320457080475000 01 -0.2528480524021900 01  0.1273088115017920-01
     0.3093844055813700 01 -0.2407788730075800 01 -0.3129576463613500 01  0.3037545099466100-01 -0.1298128184052960-01
     0.5736606038646700 01  0.1166904073419200 01 -0.3340802328411500 01  0.3148856243891200 01 -0.7484917653403920-02


S X VARIATIONAL EQUATION CHECKS -0.191490-11 0.106580-12-0.266100-11-0.116620-11 0.588420-12 0.155790-11
```

CHARACTERISTIC EXPONENTS TRACE

-0.97945064686214D 00 0.20168398637804D-00 -0.97945064686214D 00-0.20168398637804D-00 0.41098706275722D-01

SUGGESTED CHANGES IN INITIAL VALUES-0.28930321879120D-03-0.16681024179000D-03 0.16370844626180D-03-0.28372274291670D-03

SUGGESTED CHANGES IN INITIAL VALUES-0.38103026730170D-07-0.21969942568720D-07 0.21561416875800D-07-0.37368043475220D-07

NEW GUESSES

0.5040460950908060D 00 -0.8746856037638890D 00 0.8617043215148340D 00 0.4965611781723830D-00

PLANE RESTRICTED THREE BODY PROBLEM

SUN JUPITER

RM = 0.95387535710708680D-03

JACOBI CONSTANT = 0.30000732665714100D 01

PERIOD = 0.78597114868997760D 02

	X	Y	P SUB X	P SUB Y
INITIAL VALUES	0.5040461331938323D 00	-0.8746855817939466D 00	0.8617042999534174D 00	0.4965612155404261D-00
FINAL VALUES	0.5040461331944687D 00	-0.8746855817935821D 00	0.8617042999530566D 00	0.4965612155410463D-00

INITIAL DX/DT AND DY/DT VALUES WERE -0.12981281840529270D-01 -0.74849176534061980D-02

ONE OF THE CONJUGATE COMPLEX CHARACTERISTIC ROOTS IS -0.97945064686213910D 00 0.20168398637804010D-00

THE TRACE WAS 0.41098706275721990D-01